MATHEMATICS

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1962

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Abstract

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MATHEMATICS

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ON SETS ENUMERABLE AND DECIDABLE BY AUTOMATA

(Presented by Academician P. S. Novikov on 6 X 1961)

In the present note we consider the notions of enumerable and decidable sets and of a decidable predicate, analogous to those used in algorithmic set theory [1], but based not on the notion of a partial recursive function, but on the notion of an automaton with outputs.

Let $\Sigma = \{\sigma_1, \dots, \sigma_m\}$ be a fixed alphabet, and let T be the semigroup of words in this alphabet.

By an automaton (with outputs) $A = \langle S, M, s_0, F, O \rangle$ we mean the collection of the following objects: $S = \{s_0, s_1, \dots, s_{n-1}\}$—the set of internal states ($n \geqslant 1$); M—a mapping of $S \times \Sigma$ into S; $s_0 \in S$—the initial state; $F \subseteq S$—a subset of S; O—a mapping of S into T.

An automaton with outputs A defines in the following way a certain partial mapping $\mathfrak{A}$ of T into T. Let $t \in T$, with $t = \sigma_{i_1}\sigma_{i_2} \dots \sigma_{i_k}$ ($k \geqslant 1$). At the initial moment the automaton is in the state s_0. Under the action of σ_{i_1} it passes into the state $s_{j_1} = M(s_0, \sigma_{i_1})$ and prints the word $O(s_{j_1})$. Further, under the action of the next letter of the input word σ_{i_2}, it reaches the state $s_{j_2} = M(s_{j_1}, \sigma_{i_2})$ and prints the word $O(s_{j_2})$, and so on. If, when the input word has ended, the automaton has arrived at a state $s_{j_k} \in F$, then we put

$$\mathfrak{A}(t) = O(s_{j_1})\, O(s_{j_2}) \dots O(s_{j_k}).$$

In the contrary case, i.e. if $s_{j_k} \notin F$, we consider that $\mathfrak{A}(t)$ is not defined.

By the set enumerable by the automaton A we shall mean the set of values of the mapping $\mathfrak{A}$, i.e. $\mathfrak{A}(T)$. A subset $W \subseteq T$ is called finitely enumerable if it is enumerated by some automaton A. A finitely enumerable set is the analogue of a recursively enumerable set in algorithmic set theory. Alongside finitely enumerable sets one may consider strongly enumerable sets—sets which admit enumeration by everywhere-defined automata, i.e. by such automata for which $F = S$. In contrast to algorithmic theory, where the classes of sets enumerable by partial recursive and by general recursive functions coincide, here the following holds.

Theorem 1. *There exists a finitely enumerable, but not strongly enumerable, set.*

An example of such a set is the set

$$W_0 = \{\sigma_1 \underbrace{\sigma_0 \dots \sigma_0}_{n} \sigma_1\} \quad (n \geqslant 0).$$

It can easily be proved that the union and intersection of finitely enumerable sets are again finitely enumerable; however, the intersection of strongly enumerable sets may already fail to be strongly enumerable (although the union of strongly enumerable sets is strongly enumerable). An example of strongly enumerable sets whose intersection is not strongly enumerable is:

$$W_1 = \{\sigma_1 \underbrace{\sigma_0 \dots \sigma_0}_{2n},\ \sigma_1 \underbrace{\sigma_0 \dots \sigma_0}_{2n+3} \sigma_1\}, \quad W_2 = \{\sigma_1 \underbrace{\sigma_0 \dots \sigma_0}_{2n+1},\ \sigma_1 \underbrace{\sigma_0 \dots \sigma_0}_{2n+3} \sigma_1\} \quad (n \geqslant 0).$$

A subset $W \subseteq T$ is called finitely decidable if there exists an automaton A such that $t \in W$ if and only if, when the word t is fed into the automaton A, the automaton enters a state belonging to the distinguished set F. The notion of a finitely decidable set coincides with the notion of a representable event ([2]). In contrast to algorithmic theory, the following turns out to be true.

Theorem 2. *In order that a subset $W \subseteq T$ be finitely enumerable, it is necessary and sufficient that it be finitely decidable.*

The proof that a finitely enumerable set is finitely decidable is based on a theorem of Nerode stated in ([3]). In constructing a deciding automaton from an enumerating one, an exponential increase in the number of internal states occurs. V. A. Uspenskii communicated to the author the hypothesis that, for every n, there is a set enumerable by an automaton having n states, but decidable only by an automaton with $C_1 2^{C_2 n}$ states (where C_1 and C_2 are certain constants independent of n). If this hypothesis is true, then some such sets may be regarded as practically enumerable, but practically undecidable. These considerations are of interest in connection with the analysis of the abstraction of potential feasibility carried out in ([4]).

An automaton $A = \langle S, M, s_0, F, O \rangle$ defines a predicate $A(t)$, defined on the semigroup T, as follows: $A(t) \equiv M(s_0, t) \in F$. The truth set of this predicate is the set decidable by the automaton A.

A predicate defined on T for which there exists an automaton defining it will be called an automaton predicate.

Theorem 3. *Let $A_1(t)$ and $A_2(t)$ be automaton predicates. Then:*
1) $A_1(t) \& A_2(t)$; 2) $A_1(t) \vee A_2(t)$; 3) $A_1(t) \supset A_2(t)$ are automaton predicates.

Let $A(t)$ be an automaton predicate. Then: 4) $\neg A(t)$; 5a) $\exists x A(xt)$; 5b) $\exists x A(tx)$; 6a) $\forall x A(xt)$; 6b) $\forall x A(tx)$ are automaton predicates (x is a variable whose domain of admissible values is T).

It follows from Theorem 3 that, starting from automaton predicates, it is not possible to construct a hierarchy of predicates (sets) analogous to the hierarchy of Kleene–Mostowski predicates (sets) ([5]), since the quantification of automaton predicates again yields automaton predicates.

Received
5 X 1961

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5. S. C. **Kleene**, *Introduction to Metamathematics*, IL, 1957.

Note: Figure translations are in progress. See original paper for figures.