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Abstract

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A NECESSARY AND SUFFICIENT CONDITION FOR THE FREDHOLMNESS OF THE DIRICHLET PROBLEM FOR A CERTAIN CLASS OF ELLIPTIC SYSTEMS

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Consider a system of linear partial differential equations of second order of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0, \tag{1}$$

where $A = \|A_{ik}\|$, $B = \|B_{ik}\|$, $C = \|C_{ik}\|$ are given real constant square matrices of order n, and $u(x,y) = (u_1(x,y), u_2(x,y), \dots, u_n(x,y))$ is the unknown vector.

System (1) is called elliptic in the sense of Petrovskii if all $2n$ roots of the characteristic equation

$$\det |A + 2B\lambda + C\lambda^2| = 0 \tag{2}$$

are complex.

A general representation of solutions regular in a finite simply connected domain D of system (1) was obtained by A. V. Bitsadze ([1]).

In the present note we restrict ourselves to the case when (2) has one pair of complex-conjugate roots of multiplicity n. Without loss of generality one may assume that the roots of (2) are $\lambda = i$ and $\bar{\lambda} = -i$. Then the general solution of system (1), following ([1]), is written in the form

$$u(x,y) = \operatorname{Re} \sum_{l=1}^{n} \sum_{m=0}^{l-1} C_{lm} z^m \varphi_l^{(m)}(z), \tag{3}$$

where $\varphi_l(z)$ are arbitrary holomorphic functions in the domain D of the variable $z = x + iy$; the upper index m of the function $\varphi_l(z)$ denotes the order of the derivative with respect to z, and C_{lm} are constant vectors expressed exclusively in terms of the coefficients of system (1).

Following ([2]), we shall call system (1) weakly coupled if the vectors C_{l0} are linearly independent; otherwise system (1) is called strongly coupled. The C_{l0} are solutions of the following linear algebraic system of order n^2:

$$(A + 2Bi - C)C_{10} = 0,$$

$$(A + 2Bi - C)C_{20} + 2(A + C)C_{10} = 0, \tag{4}$$

$$(A + 2Bi - C)C_{k0} + 2(A + C)C_{k-1,0} + (A - 2Bi - C)C_{k-2,0} = 0,$$

$$k = 3, \dots, n.$$

All the remaining vectors C_{lk} are expressed linearly in terms of C_{l0}.

The Dirichlet problem for system (1) consists in determining a solution of this system, regular in the domain D, continuous up to the boundary Γ, and satisfying the condition

$$u = f(t) \quad \text{on } \Gamma,$$

where $f(t) = (f_1(t), \dots, f_n(t))$ is a given vector-function of a boundary point.

It is known that the Dirichlet problem for linear systems of elliptic type (in particular for system (1)), generally speaking, is not a problem of Fredholm type. The Fredholm alternative for the indicated problem always holds if the usual ellipticity requirement is strengthened by the condition of strong ellipticity (^3). However, the condition of strong ellipticity is only sufficient. There are examples of systems that are not strongly elliptic for which, nevertheless, the Dirichlet problem is always solvable and unique (^1). Therefore it is of interest to study the Dirichlet problem for elliptic systems that are not necessarily strongly elliptic.

In note (^4) the Dirichlet problem was considered in the case when $n = 2$ and the domain D is the unit disk.

In the present note we consider a bounded simply connected domain D, whose boundary Γ satisfies the Lyapunov condition, and $f(t)$ is a continuous vector-function satisfying a Hölder condition with exponent $\leqslant 1$. The number of equations n is arbitrary.

Under the assumptions made above the following theorems are proved:

Theorem 1. *If system (1) is weakly coupled, then: a) the homogeneous Dirichlet problem ($f(t) \equiv 0$) has a finite number of linearly independent solutions; b) for solvability of the nonhomogeneous Dirichlet problem it is necessary to impose on*

$f(t)$ as many additional conditions as there are solutions of the homogeneous Dirichlet problem.

Theorem 2. *Weak coupling of system (1) is a necessary and sufficient condition for the Fredholm character of the Dirichlet problem *.*

Since for system (1) the general solution (3) is known, the Dirichlet problem is nothing but a boundary value problem with respect to n unknown holomorphic functions $\varphi_1(z), \dots, \varphi_n(z)$. The most natural method for solving such a problem is to find a suitable integral representation of the unknown functions in the form of a Cauchy-type integral or analogous integrals. These expressions, substituted into the boundary conditions, give a system of integral equations; by investigating it one can draw a conclusion about the solvability of the Dirichlet problem.

In our case it is easy to show that in (3) $\varphi_l(z)$ is H_{l-1}-holomorphic**. To prove Theorem 1 the following is used:

Lemma. *Let $\varphi_l(z)$ be H_{l-1}-holomorphic in D. Then there exist a unique real-valued function $\mu_l(t)$ and a unique real constant C_l such that the representation is valid*

$$\varphi_l(z) = \frac{1}{\pi i} \int_\Gamma \frac{\mu_l(t) + \sum_{k=l+1}^{n} (-1)^{k-l} \frac{d^{\,k-l-1}}{dt^{\,k-l-1}} \left(\frac{\bar{t}^{\,k-l}}{(k-l)!} \mu_k'(t) \right)}{t-z} \, dt + iC_l, \quad (5)$$

where $\mu_l(t)$ is $(l-1)$ times differentiable and $\mu_l^{(l-1)}(t)$ satisfies a Hölder condition with exponent $\leqslant 1$.

Starting from representation (5),

$$\varphi_n(z) = \frac{1}{\pi i} \int_\Gamma \frac{\mu_n(t)}{t-z} \, dt + iC_n$$

and, regarding $\mu_n(t)$ as found, we represent $\varphi_{n-1}(z)$ as follows:

$$\varphi_{n-1}(z) = \frac{1}{\pi i} \int_\Gamma \frac{\mu_{n-1}(t) - \bar{t}\, \mu_n'(t)}{t-z} \, dt + iC_{n-1}$$

and so on.

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* Fredholmness of the Dirichlet problem is understood as formulated in Theorem 1.

** A function $\varphi(z)$ is called H_m-holomorphic in D if it is continuous and satisfies a Hölder condition with exponent $\leqslant 1$, together with its derivatives up to order m inclusive, in the closed domain $\overline{D}$.

Thus, in the representation $\varphi_l(z)$, in addition to the function $\mu_l(t)$, there enter the functions $\mu_{l+1}(t), \ldots, \mu_n(t)$, which are regarded as already known.

The Dirichlet problem is solved, with the aid of (5), by reduction to an equivalent system of singular integral equations with index $\varkappa = 0$. This completes the proof of Theorem 1.

After this, in Theorem 2 it remains only to prove that, in the case of strong connectedness of system (1), the homogeneous Dirichlet problem has infinitely many solutions, while the nonhomogeneous Dirichlet problem is unsolvable.

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Note: Figure translations are in progress. See original paper for figures.