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**Abstract**

**Full Text**

## Reports of the Academy of Sciences of the USSR

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**PHYSICS**

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### ON THE THEORY OF A NONIDEAL BOSE GAS AT A TEMPERATURE DIFFERENT FROM ZERO

*(Presented by Academician N. N. Bogolyubov, 10 XI 1961)*

In the work of N. N. Bogolyubov <sup>(1)</sup>, a theory was constructed for a nonideal Bose gas at zero temperature, which was subsequently developed in a number of works (see, for example, <sup>(2)</sup>). The method of two-time temperature functions <sup>(3,4)</sup>, used in the present work, makes it possible to carry out an investigation at temperatures different from zero.

Consider a Bose gas consisting of particles interacting pairwise with one another and described by a Hamiltonian of the form

$$H = \sum_p \left( \frac{p^2}{2m} - \mu \right) a_p^+ a_p + \frac{1}{2V} \sum_{(p_1+p_2=p'_1+p'_2)} v(p_1 - p'_1) a_{p_1}^+ a_{p_2}^+ a_{p'_2} a_{p'_1}, \quad (1)$$

where  $a_p$  and  $a_p^+$  are the Bose creation and annihilation operators for particles with momentum  $p$ ;  $\mu$  is the chemical potential;  $v(p)$  is the Fourier component of the potential of interaction of the particles.

We perform the canonical transformation of the operators

$$a_p = \varphi_p + b_p, \quad \varphi_p = \sqrt{N_0} \delta(p), \quad \delta(p) = 1, 0 \quad (p = 0, p \neq 0) \quad (2)$$

(a shift of the operators by a  $c$ -number). The quantity  $N_0$  has the meaning of the number of particles in the condensate. The quantum equation of motion will have the form

$$i \frac{db_p}{dt} = \sqrt{N_0} \left( -\mu + \frac{N_0}{V} v(0) \right) \delta(p) + \xi_p b_p + \frac{N_0}{V} v(p) b_{-p}^+ + \frac{\sqrt{N_0}}{V} \sum V_{pp_1} + \frac{1}{V} \sum W_{pp_2 p'_2 p'_1}, \quad (3)$$

where

$$\xi_p = \frac{p^2}{2m} - \mu + \frac{N_0}{V} (v(0) + v(p)),$$

$$V_{pp_1} = \{(v(p) + v(p_1))b_{-p_1}^+ + v(p_1)b_{p_1}\}b_{p-p_1},$$

$$W_{pp_2p_2'p_1'} = v(p - p_1')\delta(p + p_2 - p_1' - p_2')b_{p_2}^+ b_{p_2'} b_{p_1'}.$$

In the right-hand side of (3), the summation is carried out over all indices except  $p$ .

From the condition  $\langle b_p \rangle = \langle b_{-p}^+ \rangle = 0$  for  $p = 0$  ( $\langle \dots \rangle$  denotes averaging over the grand canonical ensemble), we obtain the expression for the chemical potential in terms of the correlation functions:

$$\begin{aligned} \mu = \frac{N_0}{V} v(0) + \frac{1}{V} \sum_{p_1} \{ & (v(0) + v(p_1)) \langle b_{p_1}^+ b_{p_1} \rangle + v(p_1) \langle b_{p_1} b_{-p_1} \rangle \} \\ & + \frac{1}{\sqrt{N_0} V} \sum_{pp_1} v(p_1) \langle b_{p_1+p}^+ b_{p_1} b_p \rangle. \end{aligned} \quad (4)$$

Introducing the matrix notation

$$\mathcal{B}_p = \begin{pmatrix} b_p \\ b_{-p}^+ \end{pmatrix}, \quad \mathcal{V}_{pp_1} = \begin{pmatrix} V_{pp_1} \\ V_{-p, -p_1}^+ \end{pmatrix}, \quad \mathcal{W}_{pp_2p_2'p_1'} = \begin{pmatrix} W_{pp_2p_2'p_1'} \\ W_{-p, -p_2, -p_2', -p_1'}^+ \end{pmatrix},$$

$$L_p = \begin{pmatrix} \xi_p & \frac{N_0}{V} v(p) \\ \frac{N_0}{V} v(p) & \xi_p \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

it is convenient to represent equation (3) and the equation conjugate to it in the form

$$\begin{aligned} \alpha i \frac{d\mathcal{B}_p}{dt} = \sqrt{N_0} \left( -\mu + \frac{N_0}{V} v(0) \right) \delta(p) + L_p \mathcal{B}_p + \\ + \frac{\sqrt{N_0}}{V} \sum \mathcal{V}_{pp_1} + \frac{1}{V} \sum \mathcal{W}_{pp_2p_2'p_1'}, \end{aligned} \quad (5)$$

We define the matrix Green function  $G_p$  as follows:

$$\begin{aligned}
 G_p(t-t') &= \langle\langle \mathcal{B}_p(t); \mathcal{B}_p^+(t') \rangle\rangle = \left\langle\left\langle \begin{pmatrix} b_p(t) \\ b_{-p}^+(t) \end{pmatrix} (b_p^+(t'), b_{-p}(t')) \right\rangle\right\rangle = \\
 &= \begin{pmatrix} \langle\langle b_p; b_p^+ \rangle\rangle & \langle\langle b_p; b_{-p} \rangle\rangle \\ \langle\langle b_{-p}^+; b_p^+ \rangle\rangle & \langle\langle b_{-p}^+; b_{-p} \rangle\rangle \end{pmatrix}, \quad (6)
 \end{aligned}$$

where, for example, for the retarded Green function  $\langle\langle b_p(t); b_p^+(t') \rangle\rangle = -i\theta(t-t')\langle[b_p(t); b_p^+(t')]\rangle$  (3,4), etc. (the arguments  $t, t'$  on the right-hand side of (6) and below are omitted for brevity of notation).

Using the equation of motion (5), passing to the Fourier components of the Green functions,

$$\langle\langle A_p(t); B_p(t') \rangle\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle\langle A_p | B_p \rangle\rangle_E e^{-iE(t-t')} dE$$

and dropping, to simplify the notation, the index  $E$ , we obtain

$$\begin{aligned}
 (\alpha E - L_p)G_p &= 1 + \frac{\sqrt{N_0}}{V} \sum \langle\langle \mathcal{V}_{pp_1} | \mathcal{B}_p^+ \rangle\rangle + \frac{1}{V} \sum \langle\langle \mathcal{W}_{pp_2p'_2p'_1} | \mathcal{B}_p^+ \rangle\rangle, \\
 \langle\langle \mathcal{V}_{pp_1} | \mathcal{B}_p^+ \rangle\rangle (\alpha E - L_p) &= \frac{\sqrt{N_0}}{V} \sum \langle\langle \mathcal{V}_{pp_1} | \mathcal{V}_{pp_1}^+ \rangle\rangle + \frac{1}{V} \sum \langle\langle \mathcal{V}_{pp_1} | \mathcal{W}_{pp_2p'_2p'_1}^+ \rangle\rangle, \\
 \langle\langle \mathcal{W}_{pp_2p'_2p'_1} | \mathcal{B}_p^+ \rangle\rangle (\alpha E - L_p) &= \langle[\mathcal{W}_{pp_2p'_2p'_1}, \mathcal{B}_p^+] \rangle \alpha + \\
 &+ \frac{\sqrt{N_0}}{V} \sum \langle\langle \mathcal{W}_{pp_2p'_2p'_1} | \mathcal{V}_{pp_1}^+ \rangle\rangle + \frac{1}{V} \sum \langle\langle \mathcal{W}_{pp_2p'_2p'_1} | \mathcal{W}_{pp_2p'_2p'_1}^+ \rangle\rangle. \quad (7)
 \end{aligned}$$

Multiplying the second and third equations (7) on the right by the matrix  $G_p^{(0)} = (\alpha E - L_p)^{-1}$ , which is the “zero approximation” to the function  $G_p$ , substituting the expressions found into the right-hand side of the first of equations (7), and multiplying this equation on the left by  $G_p^{(0)}$ , we obtain

$$G_p = G_p^{(0)} + G_p^{(0)} K_p G_p^{(0)} = G_p^{(0)} + G_p^{(0)} M_p G_p, \quad (8)$$

where  $M_p = K_p(1 + G_p^{(0)} K_p)^{-1}$  is the mass operator, and  $K_p$  is defined by relation (12) (the meaning of the parameter  $\varepsilon$  will be explained below).

Equation (8) may also be written in the form  $\Sigma_p G_p = 1$ ,  $G_p = \Sigma_p^{-1}$ , where

$$\Sigma_p = \alpha E - L_p - M_p = \alpha E - L_p - K_p(1 + G_p^{(0)} K_p)^{-1}. \quad (9)$$

The elements of the matrix  $\Sigma_p$  satisfy the following symmetry properties (5):

$$\Sigma_{22}(p, E) = \Sigma_{11}(p, -E), \quad \Sigma_{21}(p, E) = \Sigma_{12}(p, E). \quad (10)$$

Thus the problem of finding the Green functions  $G_p$  has been reduced to calculating the elements of the matrix  $\Sigma_p$ . To calculate them we shall use the following perturbation theory. We introduce a formal dimensionless small parameter  $\varepsilon$ , which in the final results we set equal to unity, and make the replacement  $v(p) \rightarrow \varepsilon v(p)$ ,  $N_0 \rightarrow \varepsilon^{-1} N_0$ . The second replacement corresponds to the assumption of a large number of particles in the condensate. Then, taking into account that  $\mathcal{V}$  and  $\mathcal{W}$  contain  $v$ , for  $K_p$  and  $\mu$  (4) we shall have

$$\begin{aligned} \mu = & \frac{N_0}{V} v(0) + \varepsilon \frac{1}{V} \sum \{ (v(0) + v(p_1)) \langle b_{p_1}^+ b_{p_1} \rangle + v(p_1) \langle b_{-p_1} b_{p_1} \rangle \} + \\ & + \frac{\varepsilon^{1/2}}{\sqrt{N_0} V} \sum v(p_1) \langle b_{p_1+p_2}^+ b_{p_1} b_{p_2} \rangle; \end{aligned} \quad (11)$$

$$\begin{aligned} K_p = & \varepsilon \frac{1}{V} \sum \{ (\nu(0) + \nu(p - p_1)) \langle b_{p_1}^+ b_{p_1} \rangle + \nu(p - p_1) \langle b_{-p_1} b_{p_1} \rangle \} + \\ & + \varepsilon \frac{N_0}{V^2} \sum \langle \mathcal{V}_{pp_1} | \mathcal{V}_{pp_1}^+ \rangle + \varepsilon^{3/2} \frac{\sqrt{N_0}}{V^2} \left\{ \sum \langle \mathcal{V}_{pp_1} | \mathcal{W}_{pp_1 p_2' p_1'}^+ \rangle + \right. \\ & \left. + \sum \langle \mathcal{W}_{pp_2 p_2' p_1'} | \mathcal{V}_{pp_1}^+ \rangle \right\} + \frac{\varepsilon^2}{V^2} \sum \langle \mathcal{W}_{pp_2 p_2' p_1'} | \mathcal{W}_{pp_2 p_2' p_1'}^+ \rangle. \end{aligned} \quad (12)$$

In the zeroth approximation  $\mu^{(0)} = \frac{N_0}{V} \nu(0)$ , the function  $G_p$  is equal to  $G_p^{(0)}$ , where

$$\xi_p \rightarrow \xi_p^{(0)} = \frac{p^2}{2m} + \frac{N_0}{V} \nu(p) \quad \text{and} \quad E_p = \sqrt{\frac{p^2}{2m} \left( \frac{p^2}{2m} + 2 \frac{N_0}{V} \nu(p) \right)}.$$

In the first approximation in  $\varepsilon$ , the matrix  $\Sigma_p$  (9) has the form

$$\Sigma_p = \alpha E - L_p - K_p, \quad (13)$$

where in  $L_p$ , in the expression for

$$\xi_p = \frac{p^2}{2m} - \mu + \frac{N_0}{V} (\nu(0) + \nu(p)),$$

one must substitute

$$\begin{aligned} \mu = \mu^{(1)} = & \frac{N_0}{V} \nu(0) + \frac{1}{V} \sum \{ (\nu(0) + \nu(p_1)) \langle b_{p_1}^+ b_{p_1} \rangle^{(0)} + \\ & + \nu(p_1) \langle b_{-p_1} b_{p_1} \rangle^{(0)} \}, \end{aligned} \quad (14)$$

$$M_p = K_p = \frac{1}{V} \sum \{(\nu(0) + \nu(p - p_1)) \langle b_{p_1}^+ b_{p_1} \rangle^{(0)} + \nu(p - p_1) \langle b_{-p_1} b_{p_1} \rangle^{(0)} \beta\} + \frac{N_0}{V^2} \sum \langle \langle \mathcal{V}_{pp_1} | \mathcal{V}_{pp_1}^+ \rangle \rangle^{(0)}. \quad (15)$$

The distribution functions  $\langle b_p^+ b_p \rangle^{(0)}$  and  $\langle b_{-p} b_p \rangle^{(0)}$  entering (14) and (15) are calculated by means of spectral representations (see <sup>(3,4)</sup>) on the basis of the Green function of the zeroth approximation  $G_p^0$  (with  $\mu = \mu^{(0)} = \frac{N_0}{V} \nu(0)$ ) and are equal to

$$\langle b_p^+ b_p \rangle^{(0)} = \frac{1}{2} \left( \frac{\xi_p^0}{E_p} \operatorname{cth} \frac{E_p}{2\theta} - 1 \right), \quad \langle b_{-p} b_p \rangle^{(0)} = -\frac{N_0 \nu(p)}{V 2E_p} \operatorname{cth} \frac{E_p}{2\theta}, \quad (16)$$

where  $\theta$  is the temperature of the Bose-particle system in energy units. The matrix elements  $\langle \langle \mathcal{V}_{pp_1} | \mathcal{V}_{pp_1}^+ \rangle \rangle$  are linear combinations of Green functions of the form  $\langle \langle b_{-p_1}^+, b_{p-p_1} | b_{p-p_1}^+ - b_{-p_1} \rangle \rangle$ ,  $\langle \langle b_p, b_{p-p_1} | b_{p-p_1}^+ - b_{-p_1} \rangle \rangle$ , etc., which in the zeroth approximation are expressed in terms of one-particle Green functions. One can also decouple the Green functions  $\langle \langle \mathcal{V}_{pp_1} | \mathcal{V}_{pp_1}^+ \rangle \rangle$  by means of Bogolyubov' s canonical  $uv$ -transformation <sup>(1)</sup>

$$b_p(t) = u_p \beta_p e^{-iE_p t} + v_p \beta_{-p}^+ e^{iE_p t}, \quad u_p^2 - v_p^2 = 1, \quad u_p^2 = \frac{1}{2} \left( 1 + \frac{\xi_p^{(0)}}{E_p} \right), \quad (17)$$

performing, in the expressions obtained after substituting (17), a pairing of the operators  $\beta_p$ :

$$\langle \beta_p^+ \beta_p \rangle^{(0)} = \frac{1}{2} \left( \operatorname{cth} \frac{E_p}{2\theta} - 1 \right), \quad \langle \beta_{-p} \beta_p \rangle^{(0)} = 0.$$

Then, for the elements of the matrix  $\Sigma_p$  (13), one obtains the expressions

$$\begin{aligned} & \frac{1}{2} (\Sigma_{11}(p, E) - \Sigma_{11}(p, -E)) = \\ & = E \left\{ 1 - \frac{N_0}{2V^2} \sum \operatorname{cth} \frac{E_{p_1}}{2\theta} \left( \frac{A_{pp_1} B_{pp_1}}{E^2 - (E_{p_1} + E_{p-p_1})^2} + \frac{C_{pp_1} D_{pp_1}}{E^2 - (E_{p_1} - E_{p-p_1})^2} \right) \right\}, \\ & -\frac{1}{2} (\Sigma_{11}(p, E) + \Sigma_{11}(p, -E)) + \Sigma_{12}(p, E) = \\ & = \frac{p^2}{2m} + \frac{1}{V} \sum (\nu(p - p_1) - \nu(p_1)) \langle b_{p_1}^+ b_{p_1} \rangle^{(0)} - \\ & -\frac{1}{V} \sum (\nu(p - p_1) + \nu(p_1)) \langle b_{-p_1} b_{p_1} \rangle^{(0)} + \\ & + \frac{N_0}{2V^2} \sum \operatorname{cth} \frac{E_{p_1}}{2\theta} \left( \frac{E_{p_1} + E_{p-p_1}}{E^2 - (E_{p_1} + E_{p-p_1})^2} B_{pp_1}^2 - \frac{E_{p_1} - E_{p-p_1}}{E^2 - (E_{p_1} - E_{p-p_1})^2} D_{pp_1}^2 \right), \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}(\Sigma_{11}(p, E) + \Sigma_{11}(p, -E)) - \Sigma_{12}(p, E) = \\
& = \frac{p^2}{2m} + \frac{1}{V} \sum (v(p-p_1) - v(p_1)) (\langle b_{p_1}^+ b_{p_1} \rangle^{(0)} + \langle b_{-p_1} b_{p_1} \rangle^{(0)}) + 2 \frac{N_0}{V} v(p) + \\
& + \frac{N_0}{2V^2} \sum \operatorname{cth} \frac{E_{p_1}}{2\theta} \left( \frac{E_{p_1} + E_{p-p_1}}{E^2 - (E_{p_1} + E_{p-p_1})^2} A_{pp_1}^2 - \frac{E_{p_1} - E_{p-p_1}}{E^2 - (E_{p_1} - E_{p-p_1})^2} C_{pp_1}^2 \right), \tag{18}
\end{aligned}$$

where

$$\begin{aligned}
A_{pp_1} &= (2v(p) + v(p_1) + v(p-p_1))(u_{p-p_1} v_{p_1} + u_{p_1} v_{p-p_1}) + \\
& \quad + (v(p_1) + v(p-p_1))(u_{p-p_1} u_{p_1} + v_{p-p_1} v_{p_1}), \\
B_{pp_1} &= (v(p_1) - v(p-p_1))(u_{p-p_1} v_{p_1} - u_{p_1} v_{p-p_1}) + \\
& \quad + (v(p_1) + v(p-p_1))(u_{p-p_1} u_{p_1} - v_{p-p_1} v_{p_1}), \\
C_{pp_1} &= (2v(p) + v(p_1) + v(p-p_1))(u_{p-p_1} u_{p_1} + v_{p-p_1} v_{p_1}) + \\
& \quad + (v(p_1) + v(p-p_1))(u_{p-p_1} v_{p_1} + u_{p_1} v_{p-p_1}), \\
D_{pp_1} &= (v(p_1) - v(p-p_1))(u_{p-p_1} u_{p_1} - v_{p-p_1} v_{p_1}) + \\
& \quad + (v(p_1) + v(p-p_1))(u_{p-p_1} v_{p_1} - u_{p_1} v_{p-p_1}). \tag{19}
\end{aligned}$$

Setting in (18)  $p = 0$ ,  $E = 0$ , and taking (16) into account, we find that

$$\begin{aligned}
& \Sigma_{11}(0, 0) - \Sigma_{12}(0, 0) = \\
& = \frac{2}{V} \sum v(p_1) \langle b_{-p_1} b_{p_1} \rangle^{(0)} - \frac{N_0}{2V^2} \sum \operatorname{cth} \frac{E_{p_1}}{2\theta} \frac{4v^2(p_1)(u_{p_1}^2 - v_{p_1}^2)^2}{-2E_{p_1}} = 0.
\end{aligned}$$

Thus, the components found for the matrix  $\Sigma_p$  satisfy the necessary condition for the absence of a gap in the spectrum of elementary excitations (see <sup>(5,6)</sup>). The energy denominators in the right-hand sides of (18) lead to the appearance of damping in the Green function  $G_p$ . At temperature  $\theta \rightarrow 0$ ,  $\operatorname{cth} \frac{E_{p_1}}{2\theta} \rightarrow 1$ , and in (18), in the last terms, the second summands under the summation signs, owing to their oddness with respect to the replacement  $p_1 \rightarrow p-p_1$ , drop out. For the phonon part of the spectrum  $E \sim p$ , and therefore for  $\theta = 0$  and  $p \rightarrow 0$  the quantity  $E$  in the denominators of expressions (18) may be

neglected. Consequently, at zero temperature the damping of the phonon part of the spectrum in the first approximation in  $p$  is absent (see (2)). Assuming that  $E \sim p$  as  $p \rightarrow 0$  and  $\theta \rightarrow 0$ , we obtain

$$G_{11}(p, E) \simeq -G_{12}(p, E) = \frac{N_0}{V} \frac{v^*(0)}{E^2 - p^2 \frac{N_0 v^*(0)}{Vm^*}},$$

where

$$v^*(0) = v(0) \left[ 1 + \frac{v(0)}{2V} \sum \left( \frac{v(p_1)}{v(0)} \right)^2 E_{p_1}^{-3} \left( \frac{p_1^2}{2m} \right)^2 \right]^{-1},$$

$$m^* = m \left[ 1 + \frac{N_0}{3V^2} \sum \frac{v^2(p_1)}{E_{p_1}^3} \frac{p_1^2}{2m} \left( 1 - \frac{p_1}{2v(p_1)} \frac{\partial v(p_1)}{\partial p_1} \right) \right]^{-1},$$

and the damping tends to zero as  $\theta \rightarrow 0$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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