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Abstract

Full Text

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ON EQUIVALENT NORMS IN THE SPACE OF FOURIER TRANSFORMS OF FINITE FUNCTIONS

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1. Let R^ν and R_ν be ν-dimensional real Euclidean spaces of vectors $x = (x^1, x^2, \dots, x^\nu)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_\nu)$; let Ω be a measurable bounded set in R^ν, $\mathrm{mes}(\Omega) > 0$. Denote by $\hat{L}_2(\Omega)$ the space of Fourier transforms $\hat{u}(\xi)$ of complex functions $u(x)$ equal to zero outside Ω and square summable, with the natural norm

$$\|\hat{u}\| = \left(\int_{R_\nu} |\hat{u}(\xi)|^2 \, d\xi\right)^{1/2} = \left(\int_\Omega |u(x)|^2 \, dx\right)^{1/2} = \|u\|. \tag{1}$$

Let M be a measurable set in R_ν, $0 < \mathrm{mes}(M) \le \infty$; $\mathfrak{M} = R_\nu \setminus M$, $E_\Omega = \{\hat{u} : \hat{u} \in \hat{L}_2(\Omega),\ \|\hat{u}\| = 1\}$. Put

$$\|\hat{u}\|_M = \left(\int_M |\hat{u}(\xi)|^2 \, d\xi\right)^{1/2}, \qquad \gamma_\Omega(\mathfrak{M}) = \sup_{\hat{u} \in E_\Omega} \int_{\mathfrak{M}} |\hat{u}(\xi)|^2 \, d\xi.$$

The norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent in $\hat{L}_2(\Omega)$ if and only if $\gamma_\Omega(\mathfrak{M}) < 1$. Following B. P. Paneiah ([1]), we shall call the set $\mathfrak{M}$ determining if $\gamma_\Omega(\mathfrak{M}) < 1$ for every bounded $\Omega \subset R^\nu$. In the work of B. P. Paneiah ([2]), for $\nu = 1$ the problem of describing determining sets is completely solved, while for $\nu > 1$ sufficient conditions are given. In the present note another approach to the problem is proposed, which for $\nu > 1$ makes it possible to obtain sufficient conditions different from those contained in ([1,2]).

2. Consider in $\hat{L}_2(\Omega)$ the bilinear functional

$$\Phi_{\mathfrak{M}}(\hat{u}, \hat{v}) = \int_{\mathfrak{M}} \hat{u}(\xi) \, \overline{\hat{v}(\xi)} \, d\xi,$$

which is evidently positive, symmetric, and bounded. It is clear that

$$\sup_{\hat{u} \in E_\Omega} \Phi_{\mathfrak{M}}(\hat{u}, \hat{u}) = \gamma_\Omega(\mathfrak{M}).$$

Consequently, there exists a positive self-adjoint bounded operator $A_{\mathfrak{M}}$, acting in $\hat{L}_2(\Omega)$, such that

$$\Phi_{\mathfrak{M}}(\hat{u}, \hat{v}) = (A_{\mathfrak{M}}\hat{u}, \hat{v}), \qquad \|A_{\mathfrak{M}}\|_{\hat{L}_2(\Omega)} = \gamma_\Omega(\mathfrak{M}).$$

It is easy to show that if F and F^{-1} are the direct and inverse Fourier operators, and $\chi_{\mathfrak{M}}$ and χ_Ω are the operators of multiplication by the characteristic functions of the sets $\mathfrak{M}$ and Ω*, then

$$A_{\mathfrak{M}}\hat{u} = F\chi_\Omega F^{-1}\chi_{\mathfrak{M}}\hat{u} = \frac{1}{(2\pi)^\nu}\int_{\mathfrak{M}} K_\Omega(\xi - \eta)\hat{u}(\eta)\, d\eta, \tag{2}$$

where

$$K_\Omega(\xi) = \int_\Omega e^{i(x,\xi)}\, d\xi. \tag{3}$$

* The operators $\chi_{\mathfrak{M}}$ and χ_Ω act, respectively, from $\hat{L}_2(\Omega)$ to $L_2(\mathfrak{M})$ and from $\hat{L}_2(\mathfrak{M})$ to $L_2(\Omega)$, where $\check{L}_2(\mathfrak{M})$ is the space of inverse Fourier transforms of functions from $L_2(\mathfrak{M})$.

Definition. We shall call a set $\mathfrak{M} \subset R_\nu$ a **set of attainability** if, for every bounded $\Omega \subset R^\nu$, there exists a function $\hat{u}_\Omega \in E_\Omega$ such that

$$\gamma_\Omega(\mathfrak{M}) = \Phi_{\mathfrak{M}}(\hat{u}_\Omega, \hat{u}_\Omega).$$

Lemma 1. Every set of attainability whose complement has positive measure is determining.

Lemma 2. In order that $\mathfrak{M}$ be a set of attainability, it is necessary and sufficient that, for every bounded $\Omega \subset R^\nu$, the operator $A_{\mathfrak{M}}$ have a maximal vector in $\hat{L}_2(\Omega)$.

Theorem 1. If, for every bounded $\Omega \subset R^\nu$,

$$\int_{\mathfrak{M}\times\mathfrak{M}} |K_\Omega(\xi - \eta)|^2\, d\xi\, d\eta < \infty, \tag{4}$$

then $\mathfrak{M}$ is a set of attainability and is determining.

Proof. Consider, along with the operator $A_{\mathfrak{M}}$, the operator $\widetilde{A}_{\mathfrak{M}} = \chi_{\mathfrak{M}}A_{\mathfrak{M}}$, acting from $\hat{L}_2(\Omega)$ into $L_2(\mathfrak{M})$. Using formula (2), the operators $A_{\mathfrak{M}}$ and $\widetilde{A}_{\mathfrak{M}}$ can be extended from $\hat{L}_2(\Omega)$ to $L_2(R_\nu)$. Here $A_{\mathfrak{M}}$ maps $L_2(R_\nu)$ into $\hat{L}_2(\Omega)$, while $\widetilde{A}_{\mathfrak{M}}$ maps $L_2(R_\nu)$ into $L_2(\mathfrak{M})$. It follows from (2) that $\widetilde{A}_{\mathfrak{M}}$ is an integral operator with kernel $\chi_{\mathfrak{M}}(\xi)K_\Omega(\xi-\eta)$, and since the range of $\widetilde{A}_{\mathfrak{M}}$ is contained in $L_2(\mathfrak{M})$, (4) implies the complete continuity of $\widetilde{A}_{\mathfrak{M}}$. It is easy to see that

$$A_{\mathfrak{M}}^2 = A_{\mathfrak{M}}\widetilde{A}_{\mathfrak{M}},$$

and hence the operator $A_{\mathfrak{M}}^2$ is completely continuous. The operator $A_{\mathfrak{M}}$ is self-adjoint on $\hat{L}_2(\Omega)$ and, consequently, is completely continuous together with $A_{\mathfrak{M}}^2$. Therefore it has a maximal vector. It remains to note that (4) implies $\operatorname{mes}(R_\nu \setminus \mathfrak{M}) = \infty$, and to use Lemmas 2 and 1.

Remark. Theorem 1 remains valid if condition (4) is replaced by the condition

$$\int_{\mathfrak{M}\times\mathfrak{M}} \prod_{i=1}^{\nu} \frac{\sin^2 \tau(\xi_i-\eta_i)}{(\xi_i-\eta_i)^2}\, d\xi\, d\eta < \infty \tag{5}$$

for some $\tau > 0$.

Let $\check{L}_2(\mathfrak{M})$ be the space of inverse Fourier transforms $\check{u}(x)$ of functions $u(\xi) \in L_2(\mathfrak{M})$, with the natural norm

$$\|\check{u}\| = \left(\int_{R^\nu} |\check{u}(x)|^2\, dx\right)^{1/2},$$

and let

$$E_{\mathfrak{M}} = \{\check{u} : \check{u} \in \check{L}_2(\mathfrak{M}),\ \|\check{u}\| = 1\}.$$

Put

$$\gamma_{\mathfrak{M}}(\Omega) = \sup_{\check{u}\in E_{\mathfrak{M}}} \int_\Omega |\check{u}(x)|^2\, dx.$$

Theorem 2. The equality

$$\gamma_{\mathfrak{M}}(\Omega) = \gamma_\Omega(\mathfrak{M})$$

holds.

Proof. Introduce the operator

$$A_\Omega = F^{-1}\chi_{\mathfrak{M}}F\chi_\Omega,$$

acting in $\check{L}_2(\mathfrak{M})$. Then the following relations are valid:

$$\gamma_{\mathfrak{M}}(\Omega) = \|F^{-1}\chi_{\mathfrak{M}}F\chi_\Omega\|_{\check{L}_2(\mathfrak{M})} \le \|\chi_{\mathfrak{M}}\|_{\hat{L}_2(\Omega)} \cdot \|\chi_\Omega\|_{\check{L}_2(\mathfrak{M})} = \sqrt{\gamma_\Omega(\mathfrak{M})} \cdot \sqrt{\gamma_{\mathfrak{M}}(\Omega)},$$

$$\gamma_\Omega(\mathfrak{M}) = \|F\chi_\Omega F^{-1}\chi_{\mathfrak{M}}\|_{\hat{L}_2(\Omega)} \le \|\chi_\Omega\|_{\check{L}_2(\mathfrak{M})} \cdot \|\chi_{\mathfrak{M}}\|_{\hat{L}_2(\Omega)} = \sqrt{\gamma_{\mathfrak{M}}(\Omega)} \cdot \sqrt{\gamma_\Omega(\mathfrak{M})}.$$

The rest is obvious.

Theorem 2 makes it possible, for every assertion about determining sets, to formulate a dual assertion. As an example, we state the following theorem, which is dual to the theorem of the paper ([2]):

Theorem 3. Let Ω be a set in R^1, not necessarily bounded, with $\beta(\Omega) < 1$*, and let $\mathfrak{M}$ be any bounded set in R_1. Then $\gamma_\Omega(\mathfrak{M}) < 1$, and consequently the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent in $\hat{L}_2(\Omega)$.

* For the definition of the quantity β see ([1,2]).

3. Here we shall give one more sufficient condition for the equivalence of the norms $\|\cdot\|_M$ and $\|\cdot\|$ in $\hat{L}_2(\Omega)$, not connected with consideration of the operator $A_{\mathfrak{M}}$.

Let $h > 0,\ a \in R_\nu,\ K_h^a = \{\xi:\ |\xi_i - a_i| \le h/2,\ i = 1, \dots, \nu\},\ K_N = \{\xi:\ |\xi_i| \le N\}$. Put

$$\alpha(\mathfrak{M}) = \lim_{h\to 0} \left\{ \lim_{N\to\infty} \left[\sup_{a\in K_N} \frac{\text{mes}(\mathfrak{M} \cap K_h^a)}{h^\nu} \right] \right\}. \tag{6}$$

It is clear that for any measurable $\mathfrak{M} \subset R_\nu$, $0 \le \alpha(\mathfrak{M}) \le 1$.

Theorem 4. If $\alpha(\mathfrak{M}) < 1$, then the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent in $\hat{L}_2(\Omega)$ for every bounded $\Omega \subset R^\nu$, and, consequently, $\mathfrak{M}$ is a determining set.

We precede the proof of Theorem 4 by two lemmas.

Lemma 3. If for all $\hat{u} \in \hat{L}_2(\Omega)$

$$\|\hat{u}\|^2 \le C_1 \int_K |\hat{u}(\xi)|^2\, d\xi + C_2 \int_M |\hat{u}(\xi)|^2\, d\xi,$$

where $K, M \subset R_\nu$, with K bounded; $\text{mes}(M) > 0$; C_1, C_2 are constants not depending on $\hat{u}$, then the norms $\|\cdot\|_M,\ \|\cdot\|$ are equivalent in $\hat{L}_2(\Omega)$.

Let $m = (m_1, m_2, \dots, m_\nu)$ be a ν-dimensional index running through all nodes of the ν-dimensional integer lattice, $\Delta_m^h = \{\xi:\ |\xi_i - m_i h| \le h/2,\ i = 1, 2, \dots, \nu\}$, $\Omega_\tau = \{x:\ |x^i| \le \tau,\ i = 1, 2, \dots, \nu\}$.

Lemma 4. There exists a positive function $C_\nu(\sigma)$, defined for $\sigma > 0$ and tending to zero together with σ, such that, when $C_\nu(\tau h) < 1$, for all $\hat{u} \in \hat{L}_2(\Omega_\tau)$ the inequality

$$\|\hat{u}\|^2 \le \frac{h^\nu}{1 - C_\nu(\tau h)} \sum_m \inf_{\xi\in\Delta_m^h} |\hat{u}(\xi)|^2$$

holds.

For example, one can show that the function $C_\nu(\sigma) = \nu^{3/2}\sigma(1+\sigma/\pi)^\nu e^{\nu\sigma}$ has all the indicated properties.

We now prove Theorem 4. Obviously, it is enough to prove that for arbitrary $\tau > 0$ the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent in $\hat{L}_2(\Omega_\tau)$. Fix an arbitrary $\tau_0 > 0$ and choose $h_0 > 0$ so small that, for $h < h_0$, the inequality $C_\nu(\tau_0 h) < 1$ holds. From the condition $\alpha(\mathfrak{M}) < 1$ it follows that there exist numbers $\delta > 0,\ h < h_0$, and $N > 0$ such that, for $a \in K_N$, the inequality $\operatorname{mes}(\mathfrak{M} \cap K_h^a) \leq h^\nu(1-\delta)$ holds. Taking into account that $M = R_\nu \setminus \mathfrak{M}$, we conclude that $\operatorname{mes}(M \cap K_h^a) \leq \delta h^\nu$ for $a \in K_N$. Let $a_m \in R_\nu$ be the point with coordinates $(m_1 h, m_2 h, \dots, m_\nu h)$, $K_m = K_h^{a_m}$. Taking into account that $\Delta_m^h = K_h^{a_m} = K_m$, and applying Lemma 4, we obtain

$$\|\hat{u}\|^2 \leq \frac{h^\nu}{1 - C_\nu(\tau_0 h)} \sum_m \inf_{\xi \in K_m} |\hat{u}(\xi)|^2 =$$

$$= \frac{h^\nu}{1 - C_\nu(\tau_0 h)} \sum_{m:\, a_m \in K_N} \inf_{K_m} |\hat{u}(\xi)|^2 + \frac{h^\nu}{1 - C_\nu(\tau_0 h)} \sum_{m:\, a_m \notin K_N} \inf_{K_m} |\hat{u}(\xi)|^2 \leq$$

$$\leq \frac{1}{1 - C_\nu(\tau_0)} \sum_{m:\, a_m \in K_N} \int_{K_m} |\hat{u}(\xi)|^2\, d\xi +$$

$$+ \frac{1}{1 - C_\nu(\tau_0 h)} \sum_{m:\, a_m \in K_N} \frac{h^\nu}{\operatorname{mes}(M \cap K_m)} \operatorname{mes}(M \cap K_m) \inf_{K_m} |\hat{u}(\xi)|^2 \leq$$

$$\leq \frac{1}{1 - C_\nu(\tau_0 h)} \int_{K_{N+h/2}} |\hat{u}(\xi)|^2\, d\xi + \frac{1}{\delta[1 - C_\nu(\tau_0 h)]} \sum_m \operatorname{mes}(M \cap K_m) \inf_{K_m} |\hat{u}(\xi)|^2 \leq$$

$$\leq \frac{1}{1 - C_\nu(\tau_0 h)} \int_{K_{N+h/2}} |\hat{u}(\xi)|^2\, d\xi + \frac{1}{\delta[1 - C_\nu(\tau_0 h)]} \int_M |\hat{u}(\xi)|^2\, d\xi,$$

and to complete the proof it remains only to use Lemma 3.

In a similar way the following theorem can be proved:

Theorem 5. *Let $\mathfrak{M}_1$ be a determining set, and let $\alpha(\mathfrak{M}_2) = 0$. Then the set $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$ is determining.*

Examples of sets $\mathfrak{M}$ for which $\alpha(\mathfrak{M}) = 0$ may be all sets of finite measure, but not only these. We note that in Theorem 5 the condition $\alpha(\mathfrak{M}_2) = 0$ cannot be replaced by the condition $\alpha(\mathfrak{M}_2) < 1$.

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REFERENCES

[1] B. P. Paneiakh, DAN, **138**, No. 1 (1961).
[2] B. P. Paneiakh, DAN, **142**, No. 5 (1962).

Note: Figure translations are in progress. See original paper for figures.