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L. B. Shneperman

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Abstract

Full Text

L. B. Shneperman

Semigroups of Continuous Transformations and Homeomorphisms of a Simple Arc

(Presented by Academician A. I. Mal' tsev on 23 V 1962)

1°. Semigroups of continuous transformations of topological spaces were studied in ⁽¹⁻³⁾. In ^(1,2), in particular, an abstract characterization was obtained of the topological semigroup of all continuous transformations of a compactum, and in ⁽³⁾ also of the algebraic semigroup for a broad class of compacta; for this purpose the corresponding semigroups are embedded in certain supersemigroups. In the present work, for the case of a simple arc, the abstract characterization can be obtained intrinsically, without embeddings.

The semigroup of homeomorphisms of a simple arc was studied in detail in ⁽⁴⁾. The fact that it does not contain the semigroup \mathfrak{H} ⁽⁶⁾ is, probably, the principal reason making it difficult to obtain its abstract characterization. Here some additional information is given, and a construction of an adjoined ideal, which make it possible to give an abstract characterization of the semigroup of homeomorphisms of a simple arc.

2°. The necessary information from the general theory of semigroups is contained in ⁽⁵⁾; notation for semigroups with topology, the definitions of an idempotent element, of an inner shift, and some others—in ⁽³⁾. We recall only that semigroups \mathfrak{A} with operation $XY = X$ ($X, Y \in \mathfrak{A}$), and only they, belong to the class of semigroups Σ .

3°. Let \mathfrak{L} be the minimal two-sided ideal of the semigroup \mathfrak{A} . Define a family Γ of subsets of the set of elements \mathfrak{L} : $\mathfrak{R} \in \Gamma$ if and only if there exist elements $L \in \mathfrak{L}$ and $A \in \mathfrak{A}$ such that $A\mathfrak{R} = L$ and $A(\mathfrak{L} \setminus \mathfrak{R}) \ni L$.

By τ we shall denote the topology on \mathfrak{L} for which the family Γ serves as a pseudobasis of closed sets.

4°. On \mathfrak{L} ^(3°) another inner topology will also be considered (i.e., a topology constructed only from the operation on the semigroup), for which the family of subsets

$$M = \{A\mathfrak{L}\}_{A \in \mathfrak{A}}$$

is taken as a pseudobasis of closed sets. We shall denote this topology by σ .

5°. Let now t be some topology on the minimal two-sided ideal \mathfrak{L} of the semigroup \mathfrak{A} , and let \mathfrak{L}_t be a compact locally connected topological space. Extend the topology t from \mathfrak{L}_t to \mathfrak{A} .

Denote by Φ the set of all regions of the topological space \mathfrak{L}_t . It is easy to show that there exists a finite system of regions covering \mathfrak{L}_t . Let $\{U_i\}$ be such a finite system. To an open neighborhood $V_{\{U_i\}}(A)$ of an element $A \in \mathfrak{A}$ we assign precisely those elements $X \in \mathfrak{A}$ for which, for every $L \in \mathfrak{L}_t$, the point XL belongs to at least one of the neighborhoods $U_i(AL)$ of the point AL . We take the set

$$\{V_{\{U_i\}}(A)\}_{A \in \mathfrak{A}, \{U_i\} \in \Phi}$$

as a pseudobasis of open sets of a topology on \mathfrak{A} . This topology will henceforth be denoted by \tilde{t} .

6°. Let Ω be a simple arc; \mathfrak{C} the semigroup of all continuous transformations of Ω ; \mathfrak{H} the subsemigroup of the semigroup \mathfrak{C} consisting of those and only those transformations which map Ω into one point;

\mathfrak{H} is the semigroup of all homeomorphisms of Ω into itself; \mathfrak{B} is the group of all homeomorphisms of Ω onto itself.

The semigroup \mathfrak{H} belongs to the class of semigroups Σ (2°) and is a minimal two-sided ideal of the semigroup \mathfrak{C} .

Let r be some fixed metric on Ω , compatible with the topology. Starting from this metric, define a metric r_0 on the semigroup \mathfrak{C} : for any $S_1, S_2 \in \mathfrak{C}$ put

$$r_0(S_1, S_2) = \max_{\xi \in \Omega} r(S_1\xi, S_2\xi).$$

The metric r_0 induces on \mathfrak{C} a topology t_0 , which does not depend on the choice of the metric r on Ω . The topological spaces \mathfrak{H}_{t_0} and Ω are homeomorphic.

7°. Let Ω_t be a minimal two-sided ideal of the topological semigroup \mathfrak{A}_t , $\Omega \in \Sigma$ (2°); Ω_t is a simple arc; the semigroup \mathfrak{A} contains no elements acting identically on the left on Ω , and on \mathfrak{A} the topologies t and \tilde{t} coincide. If r is some metric on Ω_t , compatible with the topology, then, putting

$$r(A_1, A_2) = \max_{L \in \Omega} r(A_1L, A_2L),$$

we extend it to \mathfrak{A} . In this case the metric r on \mathfrak{A}_t will be compatible with the topology. If r' is another metric on \mathfrak{A} , constructed in the same way, then the metric spaces \mathfrak{A}_r and $\mathfrak{A}_{r'}$ will be simultaneously complete or incomplete. This makes it possible to speak of the completeness of the topological space \mathfrak{A}_t , understanding by this the completeness of the metric space \mathfrak{A}_r .

On the semigroup \mathfrak{C}_{t_0} this notion of completeness coincides with the usual one.

8°. Having fixed on Ω_t (7°) one of the two possible natural orderings, the semigroup \mathfrak{A} can be ordered: for any $A_1, A_2 \in \mathfrak{A}$ put

$$A_1 \leq A_2 \leftrightarrow \forall_{L \in \Omega} A_1L \leq A_2L.$$

If, in this case, it turns out that \mathfrak{A} is a lattice (6), then under the other natural ordering of Ω_t the semigroup \mathfrak{A} also forms a lattice.

On the semigroup \mathfrak{C}_{t_0} this ordering coincides with the usual one.

9°. The results of §§ 6°–8°, (3) and one proposition from (7) (p. 36) make it possible to give an abstract characterization of the topological semigroup \mathfrak{C}_{t_0} .

Theorem. *A topological semigroup \mathfrak{A}_t is topologically isomorphic to the topological semigroup \mathfrak{C}_{t_0} if and only if:*

- 1) *the semigroup \mathfrak{A} contains a minimal two-sided ideal Ω ;*
- 2) *the semigroup Ω belongs to the class of semigroups Σ (2°);*
- 3) *the semigroup \mathfrak{A} contains no elements acting identically on Ω on the left;*
- 4) *for any four elements $L_1, L_2, L', L'' \in \Omega$, $L_1 \neq L_2$, there exists an element $A \in \mathfrak{A}$ such that $AL_1 = L'$ and $AL_2 = L''$;*
- 5) *the topological space Ω_t is a simple arc;*
- 6) *on the semigroup \mathfrak{A} the topology t coincides with the topology \tilde{t} (5°);*
- 7) *the space \mathfrak{A}_t is complete (7°);*
- 8) *the semigroup \mathfrak{A} forms a lattice (8°).*

10°. On the minimal two-sided ideal \mathfrak{H} of the semigroup \mathfrak{C} we construct the topology τ (3°) and extend it to \mathfrak{C} (5°).

Theorem. *On the semigroup \mathfrak{C} the topologies t_0 (6°) and $\tilde{\tau}$ (5°) coincide.*

11°. Theorem 10° makes it possible to assert that from the abstract characterization of the topological semigroup \mathfrak{C}_{t_0} one can always obtain an abstract characterization of the semigroup \mathfrak{C} . For this it is enough to introduce on \mathfrak{C} the topology $\tilde{\tau}$, and then to require that the resulting topological semigroup satisfy the conditions of the corresponding theorem, for example Theorem 9°.

12°. Everywhere below \mathfrak{B} is a semigroup with left cancellation, containing an identity. It is easy to show that \mathfrak{B} is a semigroup with a separating group part, i.e.

$$\mathfrak{B} = \mathfrak{G} \cup \mathfrak{A},$$

where \mathfrak{G} is the universal maximal subgroup of the semigroup \mathfrak{B} , and \mathfrak{A} is the maximal proper ideal of the semigroup \mathfrak{B} , with

$$\mathfrak{G} \cap \mathfrak{A} = \emptyset.$$

13°. Let λ be the relation of left divisibility on the semigroup \mathfrak{B} : $(X, Y) \in \lambda \leftrightarrow \exists B \in \mathfrak{B} X = YB$ for any $X, Y \in \mathfrak{B}$.

Denote by \mathfrak{P} the set of all sequences $\mathfrak{X} = \{X_1, X_2, \dots\}$ from \mathfrak{B} satisfying the conditions: a) for all n , $(X_{n+1}, X_n) \in \lambda$; b) there is no element $Z \in \mathfrak{B}$ such that, for all n , $(Z, X_n) \in \lambda$.

It is not hard to verify that

$$\{X_1, X_2, \dots\} \in \mathfrak{P} \leftrightarrow \{BX_1, BX_2, \dots\} \in \mathfrak{P}$$

for any $B \in \mathfrak{B}$.

14°. On the set $\mathfrak{D} = \mathfrak{B} \cap \mathfrak{P}$ introduce a multiplication operation: a) elements of \mathfrak{B} are multiplied as in the semigroup \mathfrak{B} ; b) if $B \in \mathfrak{B}$ and $\mathfrak{X} \in \mathfrak{P}$, then $B\mathfrak{X} = \{BX_1, BX_2, \dots\} \in \mathfrak{P}$ (13°); c) if $Q \in \mathfrak{D}$ and $\mathfrak{X} \in \mathfrak{P}$, then $\mathfrak{X}Q = \mathfrak{X}$.

It is easy to verify that \mathfrak{D} is a semigroup with respect to the multiplication operation introduced. Moreover, the subsemigroup $\mathfrak{P} \in \Sigma$ (2°) and is a minimal two-sided ideal of the semigroup \mathfrak{D} .

15°. On the semigroup \mathfrak{D} introduce a binary relation μ : for any $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{P}$,

$$(\mathfrak{X}, \mathfrak{Y}) \in \mu \leftrightarrow \forall_n (X_n, Y_n) \in \lambda.$$

Let $\bar{\mu}$ be the equivalence closure of the relation μ on \mathfrak{D} (8°).

Lemma. *The equivalence relation $\bar{\mu}$ is two-sided stable on the semigroup \mathfrak{D} .*

16°. Let φ be the natural homomorphism of the semigroup \mathfrak{D} onto the quotient semigroup $\mathfrak{D}/\bar{\mu}$. It is easy to show that, for all $B_1, B_2 \in \mathfrak{B}$,

$$\varphi(B_1) = \varphi(B_2) \leftrightarrow B_1 = B_2.$$

Therefore one may put $\varphi(B) = B$. Denoting $\varphi(\mathfrak{P}) = \mathfrak{K}$, we obtain $\varphi(\mathfrak{D}) = \mathfrak{B} \cup \mathfrak{K}$.

It follows from 14° that the subsemigroup $\mathfrak{K} \in \Sigma$ (2°) and is a minimal two-sided ideal of the semigroup \mathfrak{B} . We shall call \mathfrak{K} the adjoined ideal of the semigroup \mathfrak{B} .

17°. On the minimal two-sided ideal \mathfrak{K} of the semigroup $\mathfrak{B} \cup \mathfrak{K}$ construct the topology σ (4°).

Lemma. *If the topological space \mathfrak{K}_σ is compact, then every inner left shift of the semigroup $\mathfrak{B} \cup \mathfrak{K}$ generated by an element $B \in \mathfrak{B}$ is a homeomorphism of the space \mathfrak{K}_σ onto itself.*

18°. Let \mathfrak{K}_σ be a compact locally connected topological space. Then the topology may be extended from \mathfrak{K}_σ to \mathfrak{B} (5°). In what follows, when considering a semigroup with left cancellation that contains an identity and its subsemigroups, we shall always assume that the topology $\tilde{\sigma}$ on them is constructed precisely in this way.

19°. We pass to the study of the semigroup \mathfrak{D} (6°). Since \mathfrak{D} is a semigroup with left cancellation and contains the identity transformation, all the results of §§ 12°–18° can be carried over to it.

Moreover, note that

$$(X, Y) \in \lambda \leftrightarrow X\Omega \subset Y\Omega$$

for any $X, Y \in \mathfrak{D}$. Hence it is easy to obtain that a sequence $\mathfrak{X} = \{X_1, X_2, \dots\} \subset \mathfrak{D}$ belongs to the set of sequences \mathfrak{P} (13°) if and only if

$$X_1\Omega \supset X_2\Omega \supset \dots$$

and $\bigcap X_n\Omega$ is a point. And from this assertion it follows:

$$(\mathfrak{X}, \mathfrak{Y}) \in \mu \leftrightarrow \bigcap X_n\Omega = \bigcap Y_n\Omega$$

for any $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{P} \subset \mathfrak{D}$.

Thus, to every equivalence class of $\bar{\mu}$ (15°) on \mathfrak{P} there corresponds a point $\xi \in \Omega$, and conversely.

20°. **Theorem.** Let \mathfrak{K} be the adjoined ideal of the semigroup \mathfrak{D} (16°). The semigroups $\mathfrak{D} \cup \mathfrak{K}$ (6°) and $\mathfrak{D} \cup \mathfrak{K}$ are isomorphic.

21°. **Theorem.** On the semigroup \mathfrak{D} , the topologies $\tilde{\sigma}$ (18°) and t_0 (6°) coincide.

22°. We shall call a topological group \mathfrak{G}_t complete in itself if, for every sequence $G_1, G_2, \dots \in \mathfrak{G}_t$, from the convergence of the sequences $\{G_n G_m^{-1}\}$ and $\{G_n^{-1} G_m\}$ to the identity of the group it follows that the sequence G_1, G_2, \dots converges to $G_0 \in \mathfrak{G}_t$.

23°. Let \mathfrak{K} be the adjoined ideal of the semigroup \mathfrak{B} (16°), and let the topological space \mathfrak{K}_σ (17°) be a simple arc. Denote by K_0, K_1

its endpoints, and by \mathfrak{G}^+ the subgroup of the group \mathfrak{G} (12°) consisting of elements X such that $XK_0 = K_0$. We also note that on the simple arc \mathfrak{K}_σ we shall regard as chosen and fixed one of the two possible natural orderings.

In the following proposition an abstract characterization of the semigroup \mathfrak{D} is given.

Theorem. The semigroup \mathfrak{B} is isomorphic to the semigroup \mathfrak{D} if and only if:

- 1) \mathfrak{B} is a semigroup with left cancellation, containing an identity;
- 2) the semigroup \mathfrak{B} contains no elements acting identically from the left on its adjoined ideal \mathfrak{K} (16°);
- 3) the topological space \mathfrak{K}_σ (17°) is a simple arc;
- 4) $\mathfrak{G} \setminus \mathfrak{G}^+ \neq \emptyset$;
- 5) for any $2n$ elements $K_0 < K'_1 < \dots < K'_n < K_I$ and $K_0 < K''_1 < \dots < K''_n < K_I$ from \mathfrak{K} there exists an element $G \in \mathfrak{G}^+$ such that, for all i ,

$$GK'_i = K''_i;$$

- 6) the topological group \mathfrak{G}_σ^+ (18°) is complete in itself (22°).

24°. From Theorem 23° one easily obtains an abstract characterization of the semigroup \mathfrak{D}_{t_0} .

Theorem. The topological semigroup \mathfrak{B}_t is topologically isomorphic to the topological semigroup \mathfrak{D}_{t_0} if and only if conditions 1)-6) of Theorem 23° are satisfied and

7) on the semigroup \mathfrak{B} the topology t coincides with the topology $\tilde{\sigma}$ (18°).

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Leningrad State
Pedagogical Institute
named after A. I. Herzen

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