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# MATHEMATICS

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1962

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**Abstract**

**Full Text**

**MATHEMATICS**

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## A COMPLEX PARTIALLY ORDERED SPACE AND MODULES OVER IT

*(Presented by Academician S. L. Sobolev on 28 V 1962)*

In the paper <sup>(1)</sup>, modules over a  $K$ -space with unit <sup>(2,3)</sup>  $Z$  were constructed, generalizing the concepts of real Banach and Hilbert spaces. Here the concept of a complex partially ordered space  $V = Z \times Z$  is introduced, and modules over  $V$  are constructed that are generalizations of complex Banach and Hilbert spaces.

1. We shall agree to regard  $Z \subset \tilde{Z}$ , where  $Z$  is a  $K$ -space with unit 1,  $\tilde{Z} = C_\infty(Q)$  is its maximal extension <sup>(2,3)</sup>, which is a field <sup>(4)</sup>. Let us associate with the function  $\lambda^{-1}$  of a real argument the function  $z^{-1}$  in  $\tilde{Z}$ . Then it makes sense to consider in  $Z$  abstract functions of arguments from  $Z$ , generated <sup>(2)</sup> in  $\tilde{Z}$  by superposition of continuous functions of real arguments and functions of the form  $\lambda^{-1}$ , if the values of the abstract functions belong to  $Z$ , although certain intermediate operations are not always realizable without leaving  $Z$  (cf. Definition 1 in <sup>(1)</sup>). For example,  $\sqrt{z^2 + u^2}$  exists in  $Z$  for any  $z, u \in Z$ ; for the existence in  $Z$  of the quotient <sup>(1)</sup>  $zz'/z^2 + u^2$ , it is sufficient that  $z'/z$  exist. In establishing such facts here, the properties of the quotient <sup>(1,4)</sup>, preservation of relations for generated  $(o)$ -continuous abstract functions <sup>(2)</sup>, and the normality of the embedding of  $Z$  in  $\tilde{Z}$  <sup>(2,3)</sup> are used.
2. Suppose that in  $Z$ , for each direction <sup>(3)</sup>  $A$ , there are defined in some way nonempty classes of positive generalized sequences of elements  $\alpha$ ,  $z_\alpha \geq 0$ , converging to zero in the direction  $A$ . Put  $z_\alpha \rightarrow z$  if  $|z_\alpha - z| \rightarrow 0$ . The sequence  $z_\alpha$  is called **convergent in itself** if  $u_{\alpha\beta} = z_\alpha - z_\beta \rightarrow 0$  in the direction  $A^2$  of all pairs  $(\alpha, \beta)$ , where we put  $(\alpha, \beta) < (\gamma, \delta)$  in  $A^2$  if  $\alpha < \gamma$  and  $\beta < \delta$  in  $A$ . We shall require (cf. <sup>(1)</sup>) the fulfillment of the conditions for positive sequences:
  - a) if  $z_\alpha \rightarrow 0$  and all  $z_\alpha = z$ , then  $z = 0$ ;
  - b) if  $z_\alpha \leq u_\alpha \rightarrow 0$ , then  $z_\alpha \rightarrow 0$  (normality of convergence);
  - c) if  $z_\alpha \rightarrow 0$ ,  $u_\alpha \rightarrow 0$ , then  $z_\alpha + u_\alpha \rightarrow 0$ .

3. It follows from this that a stationary sequence  $z_\alpha = z$  converges to  $z$ ; the operations of addition and subtraction are continuous in the aggregate of both arguments; and the limit is unique.
4. Moreover, conclusions 1–3 of Theorem 2 from <sup>(5)</sup> are also valid for generalized sequences converging in the sense of § 2 of the present paper.
5. Conditions a)–c) are satisfied by all types of convergence (also for generalized sequences) indicated in § 4 of <sup>(1)</sup>, as well as by  $(t)$ - and  $(r)$ -convergence <sup>(3)</sup>;  $(r)$ -convergence with a **constant** regulator <sup>(3)</sup> 1;  $(b)$ -convergence, if  $Z$  is simultaneously a  $KN$ -linear <sup>(3)</sup>;  $(bk)$ -convergence, if  $Z$  is structurally normed by means of an Archimedean  $K$ -linear <sup>(3)</sup>  $W$  in such a way that from the comparability  $|z| \leq |u|$  in  $Z$  there follows the comparability  $\|z\| \leq \|u\|$  of the generalized norms in  $W$ .
6. These types of convergence also satisfy the following conditions. If  $z_\alpha \rightarrow 0$  in the direction  $A$ , then the sequence  $z_\beta \rightarrow 0$  in any

to a cofinal subdirection in  $A$  of elements  $\beta$ . If  $z_\delta \rightarrow 0$  along any subdirection consisting of all  $\delta > \gamma$  in  $A$ , where  $\gamma$  is any fixed element, then  $z_\alpha \rightarrow 0$  along the direction  $A$ . If  $z_\alpha \rightarrow 0$ ,  $u_\alpha \rightarrow 0$ , then  $z'_{\alpha\beta} = z_\alpha \pm u_\beta \rightarrow 0$  along the direction  $A^2$ ; hence convergence follows from convergence to itself. If  $|z_\alpha - z_\beta| \leq u_\gamma \rightarrow 0$  for all  $\alpha > \gamma$ ,  $\beta > \gamma$  in  $A$ , then  $z_\alpha$  converges to itself. The operation of multiplication by a number is continuous with respect to the totality of both factors.

7. We shall further write  $u_\alpha \downarrow u$  if  $u_\alpha \rightarrow u$  in one of the senses of item 5 and  $u_\alpha \geq u_\beta$  for all  $\alpha < \beta$  in  $A$ . If  $u_\alpha \downarrow u$ , then all  $u_\alpha \geq u$ . If  $|z_\alpha - z| \leq u_\alpha \downarrow 0$ , then  $z_\alpha \xrightarrow{(0)} z$  <sup>(3)</sup>. In particular, this also applies to monotone sequences  $\pm u_\alpha \downarrow u$ . If  $Z$  is complete (with respect to convergence), then the principle of contracting segments holds in it (segments in the sense of <sup>(3)</sup>, the sequence of segments generalized). Under certain additional conditions imposed on the sequence of products  $z_\alpha u_\alpha$ , it converges to  $\lim z_\alpha \lim u_\alpha$ .
8. Form the set  $V = Z^2$  of all pairs (in ordered notation) of elements  $z, u \in Z$ , denoting the pair by  $z + ui$ . Everywhere below  $v = z + ui$ ,  $w = z' + u'i$ ;  $\lambda, \mu$  are real numbers.

**Definition 1.** If in  $Z^2$  the operations are defined by

$$v + \lambda w = (z + \lambda z') + (u + \lambda u')i, \quad vw = (zz' - uu') + (zu' + z'u)i$$

(under the condition of existence of all products indicated in the right-hand side of the last equality), then  $V = Z^2$  will be called a **complex partially ordered space** or a  $K^2$ -**space**.

By the product  $(\lambda + \mu i)v$  we understand  $(\lambda 1 + \mu 1i)v$ .

The meaning of the embedding of the  $K^2$ -space  $V = Z^2$  into the  $K^2$ -space  $\tilde{V} = \tilde{Z}^2$  is clear;  $\tilde{V}$  is an absolute ring (the product always exists). In the  $K^2$ -space  $V$  there are preserved, evidently, the relations (cf. (2)) between the generating continuous functions of complex arguments.

If  $z \geq 0$ , then there exists a unique  $\sqrt{z+0i}$  (with positive component of the pair), and it is equal to  $\sqrt{z} + 0i$ .

Identify the set  $V_1$  of all pairs  $z+0i$  with  $Z$ , regarding  $z+0i = z$ . We introduce the notions  $\operatorname{Re} v = z$ ,  $\operatorname{Im} v = u$ ,  $|v| = \sqrt{z^2 + u^2}$ ,  $\bar{v} = z - ui$ . They make sense in  $Z$  or  $V$  for any  $v \in V$ . The following relations hold:

$$||v| - |w|| \leq |v \pm w| \leq |v| + |w|,$$

$$|z| \vee |u| \leq |v| \leq |z| + |u|, \quad |v| = \sqrt{\bar{v}v}.$$

The embedding of  $V$  into  $\tilde{V}$  is normal: if  $v \in \tilde{V}$ ,  $w \in V$ , and  $|v| \leq |w|$  in  $\tilde{Z}$ , then  $v \in V$ .

**Definition 2.** The trace  $e_v$  of an element  $v \in \tilde{Z}^2$  is the trace  $(^2, ^3) e_{|v|}$  of the element  $|v| \in Z$ . If  $|v| d |w|$   $(^2, ^3)$ , then we shall call  $v$  and  $w$  disjoint,  $v d w$ .

It follows from the definition that:  $e_v = e_z \vee e_u = e_{|z|+|u|}$ ,  $e_{v+w} \leq e_v \vee e_w$ ,  $e_{vw} = e_v \wedge e_w$ , since  $|vw| = |v||w|$  in  $\tilde{Z}$ . Therefore the following four relations are equivalent: 1)  $v d w$ ; 2)  $z d z'$  and  $u d u'$ ; 3)  $z d u'$  and  $z' d u$ ; 4)  $vw = 0$ . If  $v d w$ , then  $|v+w| = |v| + |w|$  and  $e_{v+w} = e_v \vee e_w = e_v + e_w$ .

**Definition 3.** If for  $v \in V$  there exists an element  $w \in V$  such that  $e_w = vw = e_v$ , then  $w$  will be called the inverse with respect to  $v$ ,  $w = v^{-1}$ .

From the existence of  $v^{-1}$  its uniqueness follows. In  $\tilde{V}$  every element has an inverse.

**Definition 4.** If the result  $wv^{-1}$  of operations in  $\tilde{V}$  on elements from  $V$  belongs to  $V$ , then we shall call it a quotient or fraction  $w/v$ .

According to the definition we obtain the well-known formulas:

$$|w/v| = |w|/|v|,$$

$$w/v = w\bar{v}/|v|^2 = (zz' + uu'/z^2 + u^2) + (zu' - z'u/z^2 + u^2)i$$

and so on;  $e_{w/v} = e_w \wedge e_v$ ; the relations  $v d w$  and  $w/v = 0$  are equivalent.

The  $K^2$ -space  $\tilde{V}$  is an absolute field (the quotient always exists, but  $v/v = e_v$ ). For the existence in  $V$  of the fraction  $z_1 + z_2i/u_1 + u_2i$ , it is sufficient that all

fractions  $z_m/u_n$  ( $m, n = 1, 2$ ) exist in  $Z$ . From the relations  $|v| \geq |v'|$ ,  $|w| \leq |w'|$ ,  $w'/v' \in V$ , it follows that  $w/v \in V$ . All operations defining the rings  $Z, V$  and the fields  $\tilde{Z}, \tilde{V}$  are related to one another in the usual way and to the structural relations in  $Z$  and  $\tilde{Z}$  (cf. (1–4)).

Let us define convergence in the direction  $A$  of a generalized sequence  $v_\alpha \rightarrow v$  in  $V$ , if  $|v_\alpha - v| \rightarrow 0$  in  $Z$  in the sense of item 2; the sequence  $w_\alpha$  will be called convergent in itself if  $z_{\alpha\beta} = |w_\alpha - w_\beta| \rightarrow 0$  in the direction  $A^2$ . In order that  $v_\alpha \rightarrow v$ , it is necessary and sufficient that  $\text{Re } v_\alpha \rightarrow z$ ,  $\text{Im } v_\alpha \rightarrow u$  simultaneously. Using this, it is not difficult to establish the properties of convergence in  $V$ .

9. By analogy with <sup>(1)</sup>, we introduce the notion of a generalized module  $X$  over  $V$  (the products  $vx$  exist, in general, not for all pairs of elements  $v \in V, x \in X$ ).

**Definition 5** (see Definition 2 in <sup>(1)</sup>). A generalized two-sided module  $X$  over the  $K^2$ -space  $V$ , where  $vx = xv$ ,  $1x = x$ , and from the existence of  $wx$  in  $X$  and the comparability  $|v| \leq |w|$  there follows the existence of  $vx$  in  $X$ , will be called an  $M_V$ -module.

Here equalities of the form  $v(x+y) = vx + vy$ , etc. (which define the module  $X$ ) are postulated under the condition that all products occurring in at least one part of each such equality exist. By the product  $(\lambda + \mu i)x$  we mean  $(\lambda 1 + \mu 1i)x$ .

10. By analogy with Definition 5 <sup>(1)</sup>, we introduce the notion of a  $B_V$ -module, where, in particular, the equality  $|vx| = |v||x|$  is postulated for generalized norms  $|x| \in Z_+$  (provided  $vx$  or  $v|x|$  exists). As before, we define the trace  $e_x = e_{|x|}$ , disjointness  $x d y$ , if  $|x| d |y|$ ;  $v d x$ , if  $v d |x|$ .

Let us define convergence  $x_\alpha \rightarrow x$  in  $X$  and in itself, if  $|x_\alpha - x| \rightarrow 0$  in  $Z$  in the direction  $A$ , or, respectively,  $z_{\alpha\beta} = |x_\alpha - x_\beta| \rightarrow 0$  in the direction  $A^2$  in the sense of item 2. For convergence in  $X$ , all conditions of the type listed in item 3 are fulfilled. If  $x_\alpha \rightarrow x$ , then  $|x_\alpha| \rightarrow |x|$ . If, in  $Z$ , all conditions of item 6 are also fulfilled, then similar conditions are fulfilled in  $X$  as well. We give the notions of modules of type  $B_V, B_{K^*}, B_{B_k^2}$  (cf. Definition 7 in <sup>(1)</sup>), where convergence in itself (in the direction  $A^2$ ) implies convergence (in the direction  $A$ ). In modules of type  $B_{K^*}, B_{B_k^2}$ , convergence in  $Z$  means ( $o$ )- or ( $bk$ )-convergence (see item 4 in <sup>(1)</sup>, generalized sequences).

From the identification  $V_1 = Z$  it follows that a  $B_V$ -module also has all properties of a  $B_Z$ -module <sup>(1)</sup>. In particular, a module of type  $B_{K^*}$  is a space of type  $B_K$ .

11. The definitions given in item 7 of <sup>(1)</sup> are transferred in the same way. In the definition of an  $H_V$ -module (cf. Definition 10 in <sup>(1)</sup>), instead of symmetry of the scalar product, conjugacy is postulated:  $(x, y) = \overline{(y, x)} \in V$ . In the axiomatics of an  $H_V$ -module there is the requirement  $(vx, y) = v(x, y)$ , if  $vx$  or  $v(x, y)$  exists. The well-known elementary properties hold:  $(x, vy) = \overline{v(x, y)}$ , the parallelogram formula is valid, etc. We retain the previous

definition <sup>(1)</sup> of a complete orthonormal system  $\{\varepsilon_n\}$  of an  $H_V$ -module (in particular,  $|\varepsilon_n|$  are unit elements in  $Z$ ; in general, the inequality  $|\varepsilon_n| \neq 1$  is possible).

12. Quite analogously to (1), absolute  $M_{V^-}$ ,  $B_{V^-}$ ,  $H_{V^-}$ -modules are defined (where  $Z = \tilde{Z}$  and all products  $vx$  exist); modules of type  $H_V$ ,  $H_{K^2}$ ,  $H_{B_K^2}$  (complete) are defined as special cases of modules of type  $B_V$ , etc.; we define  $M^-$ ,  $B^-$ ,  $H^-$ -extensions.

After replacing the indices  $Z, K, B_K$  in the notation for modules (for example, the index  $K$  for a module of type  $B_K$  (1)), respectively, by the indices  $V, K^2, B_K^2$ , all the assertions in §§ 5-7 of (1) remain in force (but in Theorem 7

$$|x|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2.$$

In proving the existence of an  $M$ -extension  $\tilde{X}$  for any  $M_V$ -module (cf. Theorem 1 of (1)), the product

$$\tilde{v}x = x\tilde{v} = \{v_{nx}n\}$$

is defined in  $\tilde{X}$ , where  $v_n = z_n + u_n i$ , and the cononunique  $z_n, u_n$  ( $o$ )-converge in  $\tilde{Z}$ , respectively, to  $\operatorname{Re} v, \operatorname{Im} v$ .

If in  $Z$  convergence is meant in one of the senses of § 5 of the present paper and  $v_\alpha \rightarrow v$  in  $V$ ,  $x_\alpha \rightarrow x$  in a  $B_V$ -module  $X$ , then the products  $v_\alpha x_\alpha \rightarrow vx$  in  $X$  under certain additional conditions imposed on the sequence  $v_\alpha x_\alpha$ . The same also applies to scalar products in an  $H_V$ -module.

13. In (1) and here, instead of the  $K$ -spaces  $Z$  and  $\tilde{Z}$ , one may take a  $K_\sigma$ -space with unit and its maximal extension (3).

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Received  
19 V 1962

## References

1. S. N. Slugin, *DAN*, **139**, No. 5, 1059 (1961).
2. L. V. Kantorovich, B. Z. Vulikh, A. G. Pinsker, *Functional Analysis in Semiordered Spaces*, 1950.
3. B. Z. Vulikh, *Introduction to the Theory of Semiordered Spaces*, 1961.

4. B. Z. Vulikh, *Matem. sbornik*, **22** (64), 1, 27 (1948).

5. S. N. Slugin, *DAN*, **131**, No. 6, 1261 (1960).

*Note: Figure translations are in progress. See original paper for figures.*

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