MATHEMATICS

O. S. BERLYAND, I. M. NAZAROV, A. Ya. PRESSMAN

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

O. S. BERLYAND, I. M. NAZAROV, A. Ya. PRESSMAN

i^n erfc- OR THE COMPLEX GAUSS-POISSON DISTRIBUTION

(Presented by Academician N. N. Bogolyubov, 25 VI 1962)

By the i^n erfc (en effect) or complex Gauss–Poisson distribution we shall mean the probability distribution for the occurrence of the number of events n, obeying a Poisson law whose parameter is itself a random variable following the truncated normal distribution $N(x, a, \sigma)$ with abscissa of the truncation point equal to zero:

$$P(n) = \frac{2}{1 + \operatorname{erf} \frac{a}{\sigma\sqrt{2}}} \frac{1}{\sqrt{2\pi}\sigma n!} \int_0^\infty x^n e^{-x-(x-a)^2/2\sigma^2}\, dx = \tag{1}$$

$$= \frac{e^{y^2/4-a}}{1 + \operatorname{erf} \frac{a}{y}}\, y^n i^n \operatorname{erfc}\left(\frac{y}{2} - \frac{a}{y}\right), \tag{1}$$

where $y = \sigma\sqrt{2}$.

The i^n erfc-distribution describes a sufficiently broad class of random events, for example: a) fluctuations in the number of elementary particles emitted according to a Poisson law from radiation sources whose intensity or concentration varies randomly in time or space; b) the distribution of the number of crystallization centers over the volume of a melt with a random distribution of the concentration of the substance (for example, in magmatic foci); c) the distribution of output data for discrete devices that register events obeying a Poisson law, whose counting efficiency is a random variable (in particular, for counters of various kinds of radiation operating not entirely stably), etc.

We compute the mathematical expectation and variance of the i^n erfc-distribution using the basic identity

$$\frac{e^{y^2/4-a}}{1 + \operatorname{erf} \frac{a}{y}} \sum_{n=0}^{\infty} y^n i^n \operatorname{erfc}\left(\frac{y}{2} - \frac{a}{y}\right) \equiv 1. \tag{2}$$

Rewriting (2) in the form

$$\sum_{n=0}^{\infty} y^n i_n\left(\frac{y}{2}-\frac{a}{y}\right) = e^{a-y^2/4}\left(1+\operatorname{erf}\frac{a}{y}\right) \tag{3}$$

(where, for brevity, the notation $i_n(x)$ has been used instead of $i^n \operatorname{erfc}(x)$) and differentiating (3) with respect to y, we find the first and second moments

$$M_1 = \sum_{n=1}^{\infty} nP(n) = a + \frac{y}{\sqrt{\pi}}\frac{e^{-a^2/y^2}}{1+\operatorname{erf}\frac{a}{y}}; \tag{4}$$

$$M_2 = \sum_{n=1}^{\infty} n^2 P(n) = D^2 + M_1^2 = a + a^2 + \frac{y^2}{2} + (a+1)\frac{y}{\sqrt{\pi}}\frac{e^{-a^2/y^2}}{1+\operatorname{erf}\frac{a}{y}}, \tag{5}$$

where D^2 is the variance of the distribution.

The calculation of the probabilities $P(n)$ can be carried out from tables of the functions $i^n \operatorname{erfc} x$ (1) or by the approximate formulas (2).

For $a > 10$ the i^n erfc-distribution can, with satisfactory accuracy, be approximated by the truncated normal distribution

$$\frac{2}{1+\operatorname{erf}\frac{n^*}{D\sqrt{2}}}\frac{1}{D\sqrt{2\pi}}e^{-(n-n^*)^2/2D^2}, \tag{6}$$

where n^* is the abscissa of the mode.

The position of the mode of the distribution can be determined as follows. Let $n^* = m$; then

$$y^{m-1} i_{m-1}\left(\frac{y}{2}-\frac{a}{y}\right) < y^m i_m\left(\frac{y}{2}-\frac{a}{y}\right); \tag{7}$$

$$y^{m+1} i_{m+1}\left(\frac{y}{2}-\frac{a}{y}\right) < y^m i_m\left(\frac{y}{2}-\frac{a}{y}\right), \tag{8}$$

where $\frac{y}{2}-\frac{a}{y} \geq 0$. Using the consequence of (8) $i_{m+1}(x)/i_m(x) < 1/y$ and the inequality (see (?))

$$\frac{i_{m-1}(x)}{i_m(x)} \leq 2x + \frac{m+1}{x}, \tag{9}$$

we find a lower estimate for the mode $m = n^*$ for $x = y/2 - a/y > 0$:

$$n^* > x(y-2x) - 2.$$

On the other hand, from the relations

$$\frac{d}{dx}(\ln i_m(x)) = -2x - 2(m+1)\frac{i_{m+1}(x)}{i_m(x)}, \qquad i'_m(x) = -i_{m-1}(x),$$

formula (9) and the consequence of (7) $i_m(x)/i_{m-1}(x) > 1/y$, we obtain an upper estimate for $m = n^*$ when $x > 0$, i.e. $y > \sqrt{2a}$,

$$n^* < \frac{(2x^2+1)2a}{(4x-y)y}.$$

Thus,

$$x(y-2x) - 2 < n^* < \frac{(2x^2+1)2a}{(4x-y)y}. \tag{10}$$

For $y \gg 2\sqrt{a}$ we have $n^* \simeq a$; $(a \gg 2)$.

In the case of small x, using (7) and (8), expanding the function $i^n \operatorname{erfc}(x)$ in a Taylor series and retaining two terms, we obtain

$$\frac{i_n(0) - i_{n-1}(0)x}{i_{n-1}(0) - i_{n-2}(0)x} > \frac{1}{y}, \qquad \frac{i_{n+1}(0) - i_n(0)x}{i_n(0) - i_{n-1}(0)x} < \frac{1}{y};$$

whence, a fortiori,

$$\frac{i_n(0)}{i_{n-1}(0) - i_{n-2}(0)x} > \frac{1}{y}, \qquad \frac{i_{n+1}(0) - i_n(0)x}{i_n(0)} < \frac{1}{y}.$$

Next, since

$$i_n(0) = \frac{1}{2^n\Gamma(1+n/2)},$$

we shall have

$$\frac{2\Gamma(1+(n+1)/2)}{\Gamma(1+n/2)} > \frac{y}{1+xy}, \qquad \frac{2\Gamma(1+n/2)}{\Gamma(1+(n-1)/2)} < y + nx. \tag{11}$$

Let us estimate the value of the mode of the distribution in the case of a negative argument, i.e. when $0 > -(a/y - y/2) = -x$. Using the recurrence relation

$$2mi_m(-x) = 2xi_{m-1}(-x) + i_{m-2}(-x),$$

write the ratio in the form of the continued fraction

$$\frac{i_m(-x)}{i_{m-1}(-x)} = \frac{x}{m} + \frac{1}{2m} \cdot \frac{1}{\frac{x}{m-1} + \frac{1}{2(m-1)} \cdot \frac{1}{\frac{x}{m-2} + \cdots \frac{1}{\frac{x}{2} + \frac{1}{2 \cdot 2} \cdot \frac{1}{\frac{x}{1} + \frac{1}{2}\gamma(x)}}}}. \tag{12}$$

where $\gamma(x) = i_1(-x)/i_0(-x) = 2e^{-x^2}/(\sqrt{\pi}(1 + \operatorname{erf} x))$.

From (12) it follows that, for all x,

$$\frac{1}{2k} \frac{1}{2x + \frac{(2k-2)!!}{(2k-1)!!}} \le \frac{i_{2k}(-x)}{i_{2k-1}(-x)} \le \frac{(2k-1)!!}{(2k)!!} \gamma(x) + x; \tag{13}$$

$$\frac{1}{2k+1} \frac{1}{2x + \frac{(2k-1)!! \cdot 2}{(2k)!!} \cdot \frac{1}{\gamma(x)}} \le \frac{i_{2k+1}(-x)}{i_{2k}(-x)} \le \frac{(2k)!!}{(2k+1)!!} \frac{\gamma(x)}{2} + x. \tag{14}$$

For large k, using Stirling’ s formula, we obtain

$$\frac{(2k+1)!!}{(2k)!!} = \frac{(2k+1)(2k)!}{2^{2k}(k!)^2} \sim \frac{2\sqrt{k}}{\sqrt{\pi}}; \qquad \frac{(2k)!!}{(2k-1)!!} = \frac{2^{2k}(k!)^2}{(2k)!} \sim \sqrt{\pi k}. \tag{15}$$

Without loss of generality, one may assume that $n^* = 2k$; then from (7), (8), (13), and (14) we shall have

$$\frac{\gamma(x)}{2} [y - 2x(2k+1)] \le \frac{(2k+1)!!}{(2k)!!}; \qquad \frac{(2k)!!}{(2k-1)!!} < \frac{1}{(1/y - x)\gamma(x)}, \tag{16}$$

and for large k

$$\frac{\gamma(x)\sqrt{\pi}}{4} [y - 2x(2k+1)] < \sqrt{k} < \frac{1}{(1/y - x)\gamma(x)\sqrt{\pi}}. \tag{17}$$

Formulas (16) and (17) are suitable for sufficiently small x; in particular, for $x = 0$ ($a = y^2/2$), from (16) we have

$$\frac{y}{\sqrt{\pi}} < \frac{(2k+1)!!}{(2k)!!}; \qquad \frac{(2k)!!}{(2k-1)!!} < \frac{y\sqrt{\pi}}{2}, \tag{18}$$

and for large k it follows from (18) that $\sqrt{k} = y/2$, whence

$$n^* = 2k = \frac{y^2}{2} = a. \tag{19}$$

If, however, $x = a/y - y/2$ is sufficiently large, then from (12) it follows that

$$\frac{x}{m} \le \frac{i_m(-x)}{i_{m-1}(-x)} \le \frac{x}{m} + \frac{m-1}{m}\frac{1}{2x},$$

whence, putting $n^* = m$ and using (7) and (8), we shall have

$$xy - 1 \le n^* \le \frac{x - 1/2x}{1/y - 1/2x}, \tag{20}$$

i.e., for sufficiently large x,

$$n^* \simeq xy = a - \frac{y^2}{2}.$$

Institute of Applied Geophysics
Academy of Sciences of the USSR

Received
20 VI 1962

REFERENCES

1. O. S. Berlyand, R. I. Gavrilova, A. P. Prudnikov, *Tables of integral functions, error functions and Hermite polynomials*, Minsk, 1961.

2. O. S. Berlyand, A. Ya. Pressman, DAN, 140, No. 1 (1961).

Note: Figure translations are in progress. See original paper for figures.