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Abstract

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EMBEDDING THEOREMS WITH A WEIGHT

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The embedding theorems of S. L. Sobolev ([1,2]), whose fundamental importance in the theory of boundary-value problems of mathematical physics is well known, have very many diverse generalizations, additions, and applications. Among them one should especially note the investigations of S. M. Nikol' skii ([3–6]) on the embedding of the classes $H_{p_1,\dots,p_n}^{(r_1,\dots,r_n)}[M]$, who was the first to apply, for these purposes, methods of the theory of best approximation of functions of several variables. We also note that embedding theorems with a weight were first proved by L. D. Kudryavtsev ([7,8]). Works ([9,10]) are also devoted to these questions.

In the present paper we give results of the type of S. M. Nikol' skii' s embedding theorems for integral norms containing a general weight, whose character may differ from that of the weight functions used in the above-mentioned works. These results were obtained on the basis of the theorems of ([12]), which relate the behavior of best approximations of functions of several variables in norms containing a weight, by means of entire functions of finite degrees, to the differential properties of the functions under consideration.

Below, in addition to the notation and definitions of ([12]), we shall also use the following definitions. We shall say that a function $f(x_1,\dots,x_n)$ belongs to the class $H_{p,\varphi,x_k}[\psi]$ if $\|f\|_{p,\varphi}^{(n)}<\infty$ and the inequality $\|f(x_1,\dots,x_{k-1},x_k+h,x_{k+1},\dots,x_n)-2f(x_1,\dots,x_n)+f(x_1,\dots,x_{k-1},x_k-h,x_{k+1},\dots,x_n)\|_{p,\varphi}^{(n)}\le\psi(|h|)$ holds for all h for which $\psi(|h|)$ has meaning, where $\psi(|h|)$ is a function tending to zero as $h\to 0$. In what follows we shall assume that $f(x_1,\dots,x_n)$ belongs to the class $H_{p,\varphi,n}[\psi]$ if it simultaneously belongs to all the classes $H_{p,\varphi,x_k}[\psi]$.

Let $\ln_1 x=\ln x$, $\ln_j x=\ln(\ln_{j-1}x)$ $(j=2,3,\dots)$. We shall say that a function $f(x_1,\dots,x_n)$ belongs to the class $H_{p,\varphi,x_k}^{(r)}\left[M\prod_{j=1}^{N}\ln_j^{s_j}\right]$, where $r=\bar r+\alpha>0$ ($\bar r$ is a nonnegative integer, $0<\alpha\le 1$), and s_j are arbitrary real numbers, if $f\in L_{p,\varphi}^{(n)}$ has a generalized partial derivative $\partial^{\bar r}f/\partial x_k^{\bar r}$ belonging to the class $H_{p,\varphi,x_k}[\psi]$, where

$$\psi(|h|) = M|h|^\alpha \prod_{j=1}^{N} \left(\ln_j \frac{1}{|h|}\right)^{s_j},$$

and M does not depend on h (when $s_j = 0,\ j = 1, \dots, N$, we shall simply say that $f \in H^{(r)}_{p,\varphi,x_k}[M]$).

Further we shall assume that $f(x_1, \dots, x_n)$ belongs to the class

$$H^{(r_1,\dots,r_n)}_{p,\varphi}\left[M_1 \prod_{j=1}^{N} \ln_j^{s_j^{(1)}}, \dots, M_n \prod_{j=1}^{N} \ln_j^{s_j^{(n)}}\right], \tag{1}$$

if it belongs simultaneously to the classes
$H^{(r_k)}_{p,\varphi,x_k}\left[M_k \prod_{j=1}^{N} \ln_j^{s_j^{(k)}}\right]$
$(k = 1, 2, \dots, n)$. For $\varphi \equiv 1,\ s_j^{(i)} = 0,\ (i = 1, 2, \dots, n;\ j = 1, 2, \dots, N)$

the class (1) coincides with the class $H^{(r_1,\dots,r_n)}_p[M_1, \dots, M_n]$ of S. M. Nikol' skii. Let $r_i > 0,\ 1 \le p_i \le \infty\ (i = 1, 2, \dots, n)$; we shall say that $f(x_1, \dots, x_n)$ belongs to the class

$$H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}[M], \tag{2}$$

if, in each of the variables x_i, it belongs to the class $H^{(r_i)}_{p_i,\varphi,x_i}[M]$ $(i = 1, \dots, n)$. For $\varphi \equiv 1$ the class (2) was first introduced by S. M. Nikol' skii [(6)].

Theorem 1. Let $1 \le p \le p' \le \infty,\ r_i > 0\ (i = 1, 2, \dots, n),\ 1 \le m \le n,$

$$\varkappa_m = 1 - \left(\frac{1}{p} - \frac{1}{p'}\right) \sum_{i=1}^{m} \frac{1}{r_i} - \frac{1}{p} \sum_{i=m+1}^{n} \frac{1}{r_i},$$

$$\theta_m^{(j)} = \left(\frac{1}{p} - \frac{1}{p'}\right) \sum_{i=1}^{m} \frac{s_i^{(j)}}{r_i} - \frac{1}{p} \sum_{i=m+1}^{n} \frac{s_i^{(j)}}{r_i},$$

and let f belong to the class (1).

Then:

1) If $\varkappa_m > 0$, then, for fixed $x_{m+1}, \dots, x_n$, the function f, in the variables $x_1, \dots, x_m$, belongs to the class

$$H^{(\rho_1,\dots,\rho_m)}_{p',\varphi}\left[M^* \prod_{j=1}^{N} \ln_j^{\varkappa_m s_j^{(1)} + \theta_m^{(j)}}, \dots, M^* \prod_{j=1}^{N} \ln_j^{\varkappa_m s_j^{(m)} + \theta_m^{(j)}}\right],$$

where $\rho_i = \varkappa_m r_i$, $M^* \le c_1 \left(\|f\|_{p,\varphi}^{(n)} + \sum_1^n M_k \right)$, and C_1 does not depend on f, M_k, $x_{m+1}, \dots, x_n$, h.

2) If $\varkappa_m = 0$, but there exists a natural number l such that for $j = 1, 2, \dots, l-1$ $\theta_m^{(j)} = -1$ and $\theta_m^{(l)} < -1$, then, for fixed $x_{m+1}, \dots, x_n$, the function f, with respect to $x_1, \dots, x_m$, belongs to the class

$$H_{p',\varphi,m} \left[M^* \ln_l^{\theta_m^{(l)}+1} \prod_{j=l+1}^{N} \ln_j^{\theta_m^{(j)}} \right],$$

where M^* is defined above.

In both cases

$$\|f\|_{p',\varphi}^{(m)} \le C_2 \left(\|f\|_{p,\varphi}^{(n)} + \sum_{i=1}^{n} M_i \right),$$

where C_2 does not depend on f, $x_{m+1}, \dots, x_n$.

Theorem 2. Let the function f belong to the class (1), and let $\lambda_1, \dots, \lambda_n$ be nonnegative integers,

$$\sigma = 1 - \sum_{k=1}^{n} \frac{\lambda_k}{r_k}, \qquad \Delta_j = \sum_{i=1}^{n} \frac{s_j^{(i)} \lambda_i}{r_i},$$

$$r_i' = \sigma r_i, \qquad \alpha_j^{(i)} = \sigma s_j^{(i)} + \Delta_j.$$

Then, if: 1) $\sigma > 0$, or else 2) $\sigma = 0$, but there exists a natural number l such that for $j = 1, 2, \dots, l-1$ $\Delta_j = -1$ and $\Delta_l < -1$, then on E_n there exists the partial derivative $\partial^{\lambda_1 + \dots + \lambda_n} f / \partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}$, belonging in case 1) to the class

$$H_{p,\varphi}^{(r_1', \dots, r_n')} \left[\overline{M} \prod_{j=1}^{N} \ln_j^{\alpha_j^{(1)}}, \dots, \overline{M} \prod_{j=1}^{N} \ln_j^{\alpha_j^{(n)}} \right].$$

and in case 2) to the class

$$H_{p,\varphi,n} \left[\overline{M} \ln_l^{\Delta_l+1} \prod_{j=l+1}^{N} \ln_j^{\Delta_j} \right].$$

In both cases

$$M+\left\|\frac{\partial^{\lambda_1+\cdots+\lambda_n} f}{\partial x_1^{\lambda_1}\cdots\partial x_n^{\lambda_n}}\right\|_{p,\varphi}^{(n)} \leqslant C_3\left(\|f\|_{p,\varphi}^{(n)}+\sum_{k=1}^{n} M_k\right),$$

where C_3 does not depend on f, M_k, h.

Theorem 3. Let $r_i>0$; $1\leqslant p_i\leqslant q\leqslant\infty$; n, m be natural numbers for which $1\leqslant m\leqslant n$,

$$\rho^{(i)}=\frac{r_i\varkappa}{\varkappa^{(i)}}>0 \qquad (i=1,\dots,n),$$

where

$$\varkappa=\begin{vmatrix} 1-\sum_{j=1}^{n}\frac{\frac{1}{p_j}-\frac{1}{q}}{r_j} & -\frac{1}{q}\sum_{j=1}^{n}\frac{1}{r_j} \\ -\sum_{j=m+1}^{n}\frac{\frac{1}{p_j}-\frac{1}{q}}{r_j} & 1-\frac{1}{q}\sum_{j=m+1}^{n}\frac{1}{r_j}\end{vmatrix},$$

$$\varkappa^{(i)}=1-\sum_{j=1}^{n}\frac{\frac{1}{p_j}-\frac{1}{p_i}}{r_j} \qquad (i=1,\dots,n).$$

If the function $f(x_1,\dots,x_n)$ belongs to the class $H_{p_1,\dots,p_n;\varphi}^{(r_1,\dots,r_n)}[M]$, then, for any fixed $x_{m+1},\dots,x_n$, the function f, as a function of $x_1,\dots,x_m$, belongs to the class $H_{q,\varphi}^{(\rho^{(1)},\dots,\rho^{(m)})}[\overline{M}]$, and the inequality

$$\|f\|_{q,\varphi}^{(m)}+\overline{M}<C_4\left(\min_{1\leqslant i\leqslant n}\|f\|_{p_i,\varphi}^{(n)}+M\right)$$

holds, where the constant C_4 does not depend on $f, M, x_{m+1},\dots,x_n$.

Remark. The theorems stated above, for $\varphi\equiv 1$, $s_j^{(i)}=0$ $(i=1,\dots,n;\ j=1,\dots,N)$, coincide completely with the corresponding theorems of S. M. Nikol'skii ([3-6]). For $\varphi\equiv 1$, Theorems 1 and 2 were obtained earlier in ([11]).

Let us note that if, by $B_{p,\theta,\varphi}^{(r_1,\dots,r_n)}$, we denote the space of functions $f\in L_{p,\varphi}^{(n)}$ having on E_n partial generalized unmixed derivatives in the sense of S. L. Sobolev

$$\partial^k f/\partial x_i^k\in L_{p,\varphi}^{(n)} \quad (k=0,1,\dots,r_i;\ i=1,2,\dots,n)$$

with norm

$$\|f\|_{B^{(r_1,\dots,r_n)}_{p,\theta,\varphi}} = \|f\|^{(n)}_{p,\varphi} + \sum_{i=1}^{n}\left\{\int_0^1 \frac{\omega^{\theta}_{1+[\alpha_i],x_i}\left(t;\frac{\partial^{r_i} f}{\partial x_i^{r_i}}\right)^{(n)}_{p,\varphi}}{t^{\theta\alpha_i+1}}\,dt\right\}^{1/\theta} < \infty,$$

where $r_i = \bar{r}_i + \alpha_i > 0$, $\bar{r}_i$ are nonnegative integers, $0 < \alpha_i \leqslant 1$, then

Theorems 1-3 of O. V. Besov[13] remain valid if, in their formulations, $B^{(r_1,\dots,r_n)}_{p,\theta}$ is replaced in the corresponding way by $B^{(r_1,\dots,r_n)}_{p,\theta,\varphi}$.

For $\varphi \equiv 1$, the space $B^{(r_1,\dots,r_n)}_{p,\theta,\varphi}$ coincides with O. V. Besov' s space $B^{(r_1,\dots,r_n)}_{p,\theta}$.

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Note: Figure translations are in progress. See original paper for figures.