Reports of the Academy of Sciences of the USSR

MATHEMATICAL PHYSICS

1962

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1962. Volume 144, No. 4

MATHEMATICAL PHYSICS

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ON THE LINES OF FORCE OF A MAGNETIC FIELD

(Presented by Academician N. N. Bogolyubov, 3 III 1962)

As is well known, the question of the motion of plasma in a prescribed magnetic field can, in a certain approximation, be investigated by finding the lines of force of this field. This explains the interest currently aroused by the problem of finding lines of force of magnetic fields of a definite type.

From the mathematical point of view we are confronted here with the need to study a system of three ordinary differential equations which, by virtue of the equality $\operatorname{div} H = 0$, possesses an invariant measure. Despite the efforts of many mathematicians, the study of systems of this type has not so far led, in the general case, to any satisfactory answer to the question of the behavior of the trajectories of this class of systems. There is only one case in which a satisfactory answer can be given to the question posed: this is the case of fields possessing some symmetry (in the sense of the existence of a one-parameter group of invariant transformations). In this case the system of differential equations under consideration has a first integral, owing to which the original problem reduces to the simpler problem of the arrangement of trajectories on a two-dimensional surface. Prolonged attempts by G. D. Birkhoff to obtain an analogous result in the general case were unsuccessful and, as was shown in 1954 by K. L. Siegel ([1]), could not have led to any other result.

Fig. 1

Recently, in a number of papers by A. I. Morozov, L. S. Solov' ev, and others ([2–6]), the arrangement of the lines of force of magnetic fields close to symmetric ones was investigated in considerable detail. Using the averaging method, these authors replaced the exact equations of the lines of force by simpler approximate equations, which they then studied. As a result, they found that for the

Fig. 2

Figure 2: Fig. 2

approximate equations there exist regions with different behavior of the lines of force, and found the boundary separating these regions.

However, as I recently showed ([7]), the application of the averaging method to the determination of the boundary separating regions with different behavior of the lines of force leads, generally speaking, to a qualitatively incorrect result. In reality the boundary separating regions with different behavior of the lines of force has a much more complicated form than follows from the averaged equations, for which such an arrangement of the boundary is altogether impossible. Looking ahead, I note that this unusual arrangement of the boundary separating regions with different behavior of the lines of force is one of the reasons for the appearance of protuberances on the plasma, which, as is known, lead to plasma instability.

To illustrate what has been said above, let us consider an irrotational magnetic field given by the scalar potential $H_0 z + \psi(x, y, z)$, where H_0 is con-

constant, while the function $\psi(x, y, z)$ satisfies Laplace' s equation, is periodic in z with period 2π, and

$$\int_0^{2\pi} \psi(x, y, z)\, dz = 0.$$

The equations of the lines of force of this field

$$\frac{dx}{dz} = \frac{\psi'_x(x, y, z)}{H_0 + \psi'_z(x, y, z)}, \qquad \frac{dy}{dz} = \frac{\psi'_y(x, y, z)}{H_0 + \psi'_z(x, y, z)}$$

Fig. 2

by means of a suitably chosen substitution can be transformed to the form

$$\frac{du}{dz} = \frac{1}{H_0^2} F'_v(u, v) + \frac{1}{H_0^3} f\left(u, v, z, \frac{1}{H_0}\right),$$

$$\frac{dv}{dz} = -\frac{1}{H_0^2} F'_u(u, v) + \frac{1}{H_0^3} g\left(u, v, z, \frac{1}{H_0}\right), \tag{1}$$

where

$$F(x, y) = \int_0^{2\pi} \varphi''_{xz}(x, y, z)\varphi'_y(x, y, z)\, dz = -\int_0^{2\pi} \varphi''_{yz}(x, y, z)\varphi'_x(x, y, z)\, dz,$$

and the functions $f\left(u, v, z, \frac{1}{H_0}\right)$ and $g\left(u, v, z, \frac{1}{H_0}\right)$ are periodic in z with period 2π.

The function $\varphi(x, y, z)$, encountered above, is obtained from $\psi(x, y, z)$ in the following way. Let

$$\psi(x, y, z) = \sum_{n \neq 0} \psi_n(x, y) e^{inz};$$

then

$$\varphi(x, y, z) = -i \sum_{n \neq 0} \frac{1}{n} \psi_n(x, y) e^{inz}.$$

Now let

$$\psi(x, y, z) = \sin \omega x \ \mathrm{sh}\, \lambda y \ \sin z + \cos \omega x \ \mathrm{ch}\, \lambda y \ \cos z,$$

where $\lambda^2 = \omega^2 + 1$; then

$$F(u, v) = \omega \lambda \pi \left(\mathrm{sh}^2\, \lambda v + \sin^2 \omega u\right),$$

and, consequently, the arrangement of trajectories of the system

$$\frac{du}{dz} = \frac{1}{H_0^2} F_v'(u, v), \qquad \frac{dv}{dz} = -\frac{1}{H_0^2} F_u'(u, v) \tag{2}$$

will have the form shown in Fig. 1*.

* We note in passing that, for the potential proposed in article (5),

$$\psi(x, y, z) = a_1 \,\mathrm{ch}\, q_1 y \ \sin(kz - p_1 x) + a_2 \,\mathrm{sh}\, q_2 y \ \sin(kz - p_2 x),$$

the arrangement of the trajectories of system (2), contrary to the authors' assertion, will have a form that has little in common with what is shown in Fig. 1.

On the other hand, following the ideas of work ([8]), one can define for system (1) a certain analogue of a separatrix, playing for systems of type (1) the same role as the ordinary separatrix for a system of autonomous differential equations.

Using the results of note ([7]), one can show that, in order for the arrangement of the sections by the plane $z = z_0$ of this analogue of the separatrix to coincide qualitatively with that shown in Fig. 1, it is necessary that the functions $f\left(u, v, z, \frac{1}{H_0}\right)$ and $g\left(u, v, z, \frac{1}{H_0}\right)$ satisfy a certain infinite number of functional

Fig. 3

Figure 3: Fig. 3

conditions. Direct calculation shows that, in the example we are considering, we cannot, generally speaking, satisfy even the first of these conditions. This leads to the fact that the section of the separatrix of system (1) by the plane $z = z_0$ will have the form indicated in Fig. 2. In this figure the solid line depicts the section of the separatrix of system (1), the dashed line the section of the separatrix of the system obtained from (1) by replacing z by $-z$; the arrows show in what direction the points of the sections are displaced when shifted in z by the amount $\Delta z = 2\pi$. Hence it follows that some of the solutions issuing, at $z = z_0$, from the shaded region in Fig. 2 will leave this region as z increases; moreover, among the solutions leaving the said region there exist solutions which will remain in it for an arbitrarily long time.

Fig. 3

It is not difficult to see that the fraction of solutions leaving the shaded region in Fig. 2 will depend on the magnitude of the segment AB. It can be shown that as $H_0 \to \infty$ the magnitude of the segment AB will tend to zero as $e^{-\alpha H_0^2}$, where $\alpha > 0$, and, consequently, for comparatively small values of H_0, the magnitude of the segment AB will be rather small. By choosing $\psi(x, y, z)$ in a special way, in principle one can arrange that as $H_0 \to \infty$ the magnitude of the segment AB tends to zero, for example, as $e^{-\gamma H_0^4}$, where $\gamma > 0$.

It must be noted that such an optimistic situation as in the example considered will by no means always occur. Let us consider, for example, the arrangement of the lines of force of a three-entry helical field perturbed by a corrugated field ([6]). This field can be specified by the following scalar potential:

$$\psi(r, \varphi, z) = H_0 z + \frac{h_3}{\alpha} I_3(3\alpha r) \sin 3(\varphi - \alpha z) + \frac{h_0}{\alpha} I_0(k\alpha r) \sin k\alpha z,$$

where $I_0(r)$ and $I_3(r)$ are Bessel functions of the zero and third order with imaginary argument. The equations of the lines of force of this field may be written in the following form:

$$\frac{dr}{dz} = \frac{H_r}{H_z} = \frac{\psi'_r}{\psi'_z} = \frac{3h_3 I'_3(3\alpha r)\sin 3\theta + kh_0 I'_0(k\alpha r)\sin k\alpha z}{H_0 - 3h_3 I_3(3\alpha r)\cos 3\theta + kh_0 I_0(k\alpha r)\cos k\alpha z},$$

$$\frac{d\theta}{dz} = \frac{\frac{1}{r}H_\varphi - \alpha H_z}{H_z} = \frac{\frac{1}{r^2}\psi'_\varphi - \alpha\psi'_z}{\psi'_z} =$$

$$= \frac{-\alpha H_0 + \frac{h_3}{r}\frac{\partial}{\partial r}\left(r I'_3(3\alpha r)\right)\cos 3\theta - \frac{h_0}{r}\frac{\partial}{\partial r}\left(r I'_0(k\alpha r)\right)\cos k\alpha z}{H_0 - 3h_3 I_3(3\alpha r)\cos 3\theta + kh_0 I_0(k\alpha r)\cos k\alpha z}, \qquad (3)$$

Fig. 4

Figure 4: Fig. 4

where $\theta = \varphi - \alpha z$. For $h_0 = 0$, system (3) has the first integral

$$\frac{H_0 \alpha r^2}{2} - r h_3 I_3'(3\alpha r)\cos 3\theta = \text{const},$$

according to which the disposition of the trajectories of system (3) for $h_0 = 0$ has the form shown in Fig. 3.

On the other hand, using the results of note ([9]), it is not difficult to establish that the section of the separatrix of system (3) by the plane $z = z_0$ has the form shown in Fig. 4; moreover, as $h_0/H_0 \to 0$, the magnitude of the segment CD will tend to zero as the first power of the quantity h_0/H_0 (here it is assumed that the ratio h_3/H_0 is fixed).

Fig. 4

Here, just as in the preceding example, among the solutions leaving the region shaded in Fig. 4 there exist solutions that will remain in the named region for an arbitrarily long time. Therefore an attempt to find, by numerical integration of system (3) over a small interval of variation of z, the boundary separating the solutions of system (3) that will remain for a long time in the region shaded in Fig. 4 from the solutions leaving this region can hardly give a satisfactory result, since it is far from clear whether all solutions of system (3) issuing from petal-shaped regions will remain in the named region ([5,6]).

It should be noted that, on the basis of the results of note ([9]), one can construct examples of potentials analogous to the one considered above, for which the magnitude of the segment CD will have any prescribed order relative to the perturbation parameter. The choice of such fields for confining plasma may substantially increase the lifetime of the plasma.

In conclusion I take the opportunity to express my gratitude to S. V. Fomin for his interest in this work.

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for Nuclear Research

Received
17 II 1962

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Note: Figure translations are in progress. See original paper for figures.