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Abstract

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MATHEMATICS

N. N. URAL' TSEVA

BOUNDARY-VALUE PROBLEMS FOR QUASI-LINEAR ELLIPTIC EQUATIONS AND SYSTEMS WITH DIVERGENT PRINCIPAL PART

(Presented by Academician V. I. Smirnov on 4 VI 1962)

In papers (¹) the first boundary-value problem was studied for quasilinear elliptic equations with divergent principal part

$$Lu \equiv \frac{\partial}{\partial x_i} (a_i(x, u, u_{x_k})) + a(x, u, u_{x_k}) = 0. \quad (1)$$

In the present paper, for equations (1) we consider the case of boundary conditions of the form

$$L^{(S)}u \equiv [a_i(x, u, u_{x_k}) \cos(n, x_i) + \varphi(x, u)]|_S = 0. \quad (2)$$

Here S is the boundary of the domain Ω in which $x = x(x_1, \dots, x_n)$ varies; $n \geq 2$; n is the exterior normal to S . It is assumed that S belongs to C_2 .* Problem (1), (2) is a generalization of the second and third boundary-value problems to the case of quasilinear equations (1). Under natural restrictions on the functions $a_i(x, u, p_k)$ and $a(x, u, p_k)$ we have obtained a priori estimates needed for proving the existence of classical solutions of problem (1), (2), and have proved existence theorems.

Analogous results have also been established by us for quasilinear elliptic systems of the form

$$\mathcal{L}u \equiv a_{ij}(x, u)u_{x_i x_j} + b_i(x, u, u_{x_k})u_{x_i} + a(x, u, u_{x_k}) = 0. \quad (3)$$

under the boundary conditions:

$$\mathcal{L}^{(S)}u \equiv [a_{ij}(x, u)u_{x_j} \cos(n, x_i) + \vec{\varphi}(x, u)]|_S = 0, \quad (4)$$

where $\mathbf{u} = (u^1, \dots, u^N)$, $\mathbf{a} = (a^1, \dots, a^N)$, $\vec{\varphi}(\varphi^1, \dots, \varphi^N)$, and b_i and \mathbf{a} satisfy the conditions (9), (10) formulated below. The case of the first boundary-value problem for systems (3) was considered in (2).

We shall assume that the function $\varphi(x, u)$ in condition (2), for $x \in S$, is the boundary value of some function $\varphi(x, u)$ defined for $x \in \Omega$ and arbitrary u . The same applies to $\vec{\varphi}(x, u)$ from (4).

Theorem 1. Let $u(x)$ be a solution of problem (1), (2) from $C_2(\bar{\Omega})$, and let

$$\max_{\Omega} |u(x)| \leq M.$$

Suppose that for $x \in \Omega$, $|u| \leq M$, and arbitrary p_k , the following conditions are satisfied:

* For the definition of the classes C_l and $C_{l,\alpha}$, see, for example, the survey (1a).

1) Equation (1) is uniformly elliptic

$$\nu(1 + \rho)^{m-2} \sum_{i=1}^n \xi_i^2 \leq \frac{\partial a_i(x, u, p_k)}{\partial p_j} \xi_i \xi_j \leq \mu(1 + \rho)^{m-2} \sum_{i=1}^n \xi_i^2,$$

$$\nu, \mu = \text{const} > 0; \quad m = \text{const} > 1.$$

2) The functions $a_i(x, u, p_k)$ and $a(x, u, p_k)$ are differentiable with respect to x_k, u, p_k and satisfy the natural restrictions on the consistency of orders of growth with respect to

$$\rho = \left[\sum_{k=1}^n p_k^2 \right]^{1/2} :$$

$$\left| \frac{\partial a_i}{\partial p_j} \right| (1 + \rho)^2 + \left[|a_i| + \left| \frac{\partial a_i}{\partial u} \right| + \left| \frac{\partial a_i}{\partial x_k} \right| + \left| \frac{\partial a}{\partial p_k} \right| \right] (1 + \rho) +$$

$$+ |a| + \left| \frac{\partial a}{\partial u} \right| + \left| \frac{\partial a}{\partial x_k} \right| \leq \mu(1 + \rho)^m.$$

3) $\varphi(x, u)$ is twice differentiable with respect to x_k, u , and the maxima of the moduli of φ and of its partial derivatives up to the second order are bounded by the constant μ . Then $|u|_{C_{1,\alpha}(\Omega)}$, for some $\alpha > 0$, is estimated in terms of the constants ν, μ, m from conditions 1)–3), M, n , and the norm in C_2 of the boundary S .

Let us note that for any interior subdomain Ω' of the domain Ω , in the works (1) the norm $|u|_{C_{1,\alpha}(\Omega')}$ has already been estimated—it is determined by the quantity M , the functions a_i and a , and the distance from Ω' to S ; hence it remains for us only to obtain an estimate of the norm of u in $C_{1,\alpha}$ near the boundary S . For this purpose, by a transformation of the independent variables $y_i = y_i(x)$, $i = 1, \dots, n$, we carry some piece S_1 of the boundary S into a piece of the plane $y_n = 0$. Let $\eta(y)$ be an arbitrary function from $W_2^1(\Omega)$, equal to zero on $S - S_1$. We multiply equation (1) by η and integrate over Ω . Then, in the first term, we integrate by parts and in the boundary integral use condition (2). After this, transforming the boundary integral again into a volume integral, we arrive at the integral identity

$$\int_{\Omega} \left[a_i \eta_{y_k} \frac{\partial y_k}{\partial x_i} + a_i \frac{\partial^2 y_k}{\partial y_k \partial x_i} \eta - a \eta + \frac{\partial}{\partial y_n} (\psi \eta) \right] dy = 0, \quad (5)$$

where

$$\psi(y, u) = \varphi(x, u) \left[\sum_{k=1}^n \left(\frac{\partial y_n}{\partial x_k} \right)^2 \right]^{1/2}.$$

From (5), taking into account conditions 1)–3), we infer that $u(x)$ near S_1 satisfies the integral inequalities

$$\int_{\{u > \lambda\}} |\nabla u|^m \xi^m dx \leq \gamma \int_{\{u > \lambda\}} [(u - \lambda)^m |\nabla \xi|^m + \xi^m] dy, \quad (6)$$

in which $\xi(y)$ is an arbitrary nonnegative smooth function equal to zero outside some ball $K(\rho)$ of radius ρ with center on S_1 , and $\lambda \geq \max_{K(\rho)} u - \delta$. Here $\delta > 0$ and γ are certain constants. Analogous inequalities (6') are valid for the sets $\{u < \lambda\}$ when $\lambda \leq \min_{K(\rho)} u + \delta$. Inequalities (6) and (6') make it possible to estimate $|u|_{C_{0,\beta}}$ near S_1 in terms of M . Somewhat more complicated is the estimate of

$$|u_{y_k}|_{C_{0,\alpha}}, \quad k = 1, \dots, n,$$

in terms of

$$\max_{\Omega} |\nabla u|.$$

The estimate of $\max |\nabla u|$ near S_1 is carried out integrally; it is somewhat reminiscent of the estimate of $\max |\nabla u|$ from papers (16,), but is more complicated.

First of all, the integrals are estimated successively

$$\int_{\Omega} \left(1 + \sum_{k \neq n} u_{y_k}^2\right)^r (1 + |\nabla u|^2)^q \left(1 + \sum_{k \neq n} u_{y_k y_i}^2\right) \zeta^2 dy \leq C(r, q),$$

first for $q = 0$, $r = 0, 1, 2, \dots$, and then also for $q = 1, 2, \dots$, where $C(r, q)$ grows with r and q ; for what follows it suffices to take r and q large, but finite.

After this one establishes the estimate $\max |u_{y_k}|$, $k = 1, \dots, n - 1$, near S_1 , analogously to how $\max |\nabla u|$ was estimated in the interior subdomain Ω' in (16,). The derivative u_{y_n} on S_1 can be expressed through the remaining u_{y_k} from the boundary condition (2). Knowing the estimate of $\max_{S_1} |\nabla u|$, one can estimate $\max |\nabla u|$ near S_1 in the same way as is done in the case of the first boundary-value problem (16).

Theorem 1 and Schauder-type estimates for solutions of linear equations, in particular the estimates of Fiorenza (3) for the problem with oblique derivative, make it possible to prove existence theorems for problem (1)–(2).

We formulate one of them, which follows directly from the estimates obtained and from the conditions of Fiorenza's theorem (3) concerning quasilinear equations. To this end consider a family of boundary-value problems depending on the parameter $\tau \in [0, 1]$:

$$L_{\tau} u \equiv \tau L u + (1 - \tau) L_0 u = 0; \quad (7)$$

$$L_{\tau}^{(S)} u \equiv \tau L^{(S)} u + (1 - \tau) L_0^{(S)} u, \quad (8)$$

where

$$L_0 u \equiv \frac{\partial}{\partial x_i} (F_{u_{x_i}}^0(u, u_{x_k})) - F_u^0(u, u_{x_k}),$$

$$F^0(u, p_k) = (1 + p^2)^{m/2} + u^2, \quad L_0^{(S)} u \equiv F_{u_{x_i}}^0(u, u_{x_k}) \cos(n, x_i) \Big|_S.$$

Theorem 2. There exists a unique solution $u(x)$ of problem (1)–(2), belonging to $C_{2,\alpha}(\bar{\Omega})$, if the following assumptions are fulfilled:

- a) for $x \in \bar{\Omega}$ and arbitrary u, p_k , the functions $a_i(x, u, p_k)$, $a(x, u, p_k)$ satisfy conditions 1)–2) of Theorem 1, in which ν is a positive nondecreasing function and μ is a positive nonincreasing function of $|u|$;
- b) for the same values of x, u, p_k , the functions $a_i(x, u, p_k)$, $a(x, u, p_k)$, $\varphi(x, u)$ belong to the classes C_3 , C_2 , and $C_{2,\alpha}$, respectively, in all their arguments;
- c) S belongs to $C_{2,\alpha}$;
- d) one can give an a priori estimate $\max_{\Omega} |u(x, \tau)|$ for the solutions of the boundary-value problems (7), (8), uniformly with respect to $\tau \in [0, 1]$;

e) for every solution of problem (7), (8) belonging to $C_{2,\alpha}(\overline{\Omega})$, the corresponding problem in variations is unrestrictedly solvable.

Clarifying the conditions under which requirements d) and e) are fulfilled is, in the general case, connected with the investigation of the spectrum of the operator L on functions satisfying conditions (2). There are sufficient conditions under which the assumptions of points d) and e) are valid.

Consider quasilinear elliptic systems (3) under boundary conditions (4). Suppose that, for $x \in \Omega$ and arbitrary u^l, p_k^l , the functions a_{ij} are differentiable with respect to x_k, u^l , and b_i and a satisfy the following conditions:

$$|b_i(x, u, p_k)| \leq \mu(|u|)(1 + p); \quad (9)$$

$$|a(x, u, p_k)| \leq \mu(|u|)(\varepsilon + P(p)), \quad (10)$$

where

$$p = \left[\sum_{k=1}^n \sum_{l=1}^N (p_k^l)^2 \right]^{1/2}, \quad p^{-2}P(p) \rightarrow 0 \quad \text{as } p \rightarrow \infty,$$

and $\varepsilon > 0$ is some sufficiently small number.

Suppose the ellipticity condition is satisfied

$$a_{ij}(x, u)\xi_i\xi_j \geq \nu(|u|) \sum_{i=1}^n \xi_i^2, \quad \nu > 0. \quad (11)$$

Assume that S belongs to $C_{2,\alpha}$, and that $\tilde{\varphi}(x, u)$ is twice continuously differentiable. Under these conditions the following is true.

Theorem 3. *Suppose it is possible to give, uniformly with respect to τ in $[0, 1]$, an a priori estimate $\max_{\Omega} |u(x, \tau)|$ for the solutions $u(x, \tau)$ of the boundary-value problems*

$$\mathcal{L}_{\tau}u \equiv \tau\mathcal{L}u + (1 - \tau)(\Delta u - u) = 0;$$

$$\mathcal{L}_{\tau}^S u \equiv \tau\mathcal{L}^{(S)}u + (1 - \tau) \frac{\partial u}{\partial n} \Big|_S = 0.$$

Then there exists a solution $u(x)$ of problem (3), (4), belonging to the class $C_{2,\alpha}(\overline{\Omega})$.

Leningrad State University
named after A. A. Zhdanov

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¹ O. A. Ladyzhenskaya, N. N. Ural' tseva, a) UMN, **16**, 1 (1961); b) DAN, **140**, No. 1 (1961); c) O. A. Ladyzhenskaya, N. N. Ural' tseva, Comm. Pure and Appl. Math., **14**, No. 3 (1961). ² N. N. Ural' tseva, DAN, **146**, No. 4 (1962). ³ R. Fiorenza, Ric. Mat., **8**, No. 1, 83 (1959).

Note: Figure translations are in progress. See original paper for figures.

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