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Abstract

Full Text

MATHEMATICS

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ON THE EXISTENCE AND UNIQUENESS OF A SOLUTION IN THE WHOLE PLANE OF CERTAIN ELLIPTIC EQUATIONS

(Presented by Academician I. G. Petrovsky on 26 VII 1961)

§ 1. **Introduction.** In this paper we investigate the question of the existence and uniqueness of a solution of the equation

$$P\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right)u = f, \tag{1}$$

given in the whole plane, where $P\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right)$ is an elliptic operator with constant coefficients.

Let W denote the class of functions in which there exists a unique solution of equation (1).

For the equation $(\Delta + k^2)u = f$, where Δ is the Laplace operator in two or three variables, k is a real constant, the asymptotic conditions selecting the class W (radiation conditions) were found by Sommerfeld (^{1}). When our operator is metaharmonic without multiple roots in a space with an arbitrary number of variables, i.e. is an operator of the form $P(\Delta)$, where $P(s)$ is a polynomial having no multiple roots, the class W was found by I. N. Vekua (^{2}). In the work of B. P. Paneah (^{3}), conditions selecting the class W for any metaharmonic operator were obtained. When the characteristic polynomial $P(s_1, s_2)$ does not vanish in the real plane, the class W was found by V. P. Palamodov (^{4}).

In the present paper we consider the case where

$$P\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right)u \equiv \left[(-1)^n\frac{\partial^{2n}}{\partial x^{2n}} + (-1)^n\frac{\partial^{2n}}{\partial y^{2n}} - k^{2n}\right]u = f, \tag{2}$$

but the method by which the class W is obtained for equation (2) is suitable for a much wider class of equations in two variables.

In order to find the class W, a special fundamental solution is constructed and its asymptotics are found as $r = \sqrt{x^2 + y^2} \to \infty$.

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§ 2. **Construction of the fundamental solution.** The characteristic polynomial for the operator (2) is

$$P(s_1, s_2) = s_1^{2n} + s_2^{2n} - k^{2n}. \tag{3}$$

Let $s_1 = \sigma_1 + i\tau_1,\ s_2 = \sigma_2 + i\tau_2$. In the real plane the zeros of the characteristic polynomial form one oval: $\sigma_1^{2n} + \sigma_2^{2n} - k^{2n}$.

In three-dimensional space $(\sigma_1, \sigma_2, \tau_1)$ construct the following set $H(a)$: for $|\sigma_2| \geq k$ the set $H(a)$ coincides with the plane $\tau_1 = 0$; for $|\sigma_2| < k$ and $|\sigma_1| < k$ the set $H(a)$ coincides with the plane $\tau_1 + \alpha\sigma_1 = 0$; for $|\sigma_2| < k$ and $|\sigma_1| \geq k$ the set $H(a)$ coincides with the planes $\tau = \pm\alpha k$, respectively for $\sigma_1 < -k$ and $\sigma_1 > k$.

Here $|\alpha|$ is chosen so small that the roots of the polynomial $P(s_1, \sigma_2)$ with different signs of $\operatorname{Im} s_1$ lie on different sides of $H(a)$. This can always be done. Denote by H_1 the set $H(a)$ for the above-chosen a, if $\alpha > 0$, and by H_2 the corresponding set for $\alpha < 0$.

The set $H_i,\ i = 1, 2$, is not a Hermander ladder (^{5}), since $H_i,\ i = 1, 2$, contains zeros of the characteristic polynomial. Nevertheless, the following theorem holds:

Theorem 1. The integrals

$$E_i(x, y) = \iint_{H_i} \frac{e^{ixs_1 + iys_2}}{P(s_1, s_2)}\, dH_i = \iint_{H_i} \frac{e^{ixs_1 + iys_2}}{s_1^{2n} + s_2^{2n} - k^{2n}}\, dH_i, \qquad i = 1, 2, \tag{4}$$

exist and give two fundamental solutions for the operator (2).

Theorem 2. If one considers the operators

$$P_\varepsilon\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right) = P\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right) + i\varepsilon, \tag{5}$$

then their fundamental solutions, which are given by the formulas

$$E_\varepsilon(x, y) = \int_{-\infty}^{\infty}\!\!\int \frac{e^{ixs_1 + iys_2}}{P(\sigma_1, \sigma_2) + i\varepsilon}\, d\sigma_1\, d\sigma_2, \tag{6}$$

will converge, as $\varepsilon \to -0$, to $E_1(x, y)$, and, as $\varepsilon \to +0$, to $E_2(x, y)$.

Hence, in particular, it follows that if, instead of $H_i,\ i = 1, 2$, one takes analogous manifolds in the space $(\sigma_1, \sigma_2, \tau_2)$ and with their help constructs fundamental solutions for equation (2), then one obtains the very same fundamental solutions $E_1(x, y)$ and $E_2(x, y)$.

§ 3. **Asymptotics of the fundamental solutions.** In order to obtain the asymptotics of the fundamental solutions found, we shall compute the integral (4) in the following way. In integrating with respect to the variable s_1 we shall use the residue theorem, and then, in integrating with respect to σ_2, we shall apply the saddle-point method. Here the greatest difficulty is the estimation of the remainder term in a neighborhood of the directions $\varphi = k\pi/2,\ k = 0, 1, 2, 3$.

Let $\theta_1 = \cos\left(\arctan \frac{y}{x}\right),\ \theta_2 = \sin\left(\arctan \frac{y}{x}\right)$; then for the derivatives of the fundamental solutions we have:

Theorem 3.

$$\frac{\partial^{p+l} E_1(x,y)}{\partial x^p \partial y^l} =$$

$$= C_k (ik)^{p+l} \frac{\exp\left[ik\left(\theta_1^{\frac{2n}{2n-1}} + \theta_2^{\frac{2n}{2n-1}}\right)^{\frac{2n-1}{2n}} r\right]}{\sqrt{r}} \frac{\left(\theta_1^{\frac{2n}{2n-1}} + \theta_2^{\frac{2n}{2n-1}}\right)^{\frac{2n-3-2(p+l)}{4n}}}{\theta_1^{\frac{n-1-p}{2n-1}}\ \theta_2^{\frac{n-1-l}{2n-1}}} + \omega_{pl}, \tag{7}$$

where

$$|\omega_{pl}| < \frac{c\,|\theta_1|^{\frac{p+1}{2n-1}}\,|\theta_2|^{\frac{l+1}{2n-1}}}{\left(r\,|\theta_1\theta_2|^{\frac{2n}{2n-1}}\right)^{\gamma}}, \qquad 1 > \gamma \geq \frac{1}{2} \tag{8}$$

for all γ in the indicated interval.

For $E_2(x,y)$ one obtains the same formulas, only in (7) one must take $-k$ instead of k.

§ 4. **The class W.** Formulas (7) and (8) make it possible to obtain conditions at infinity which ensure the existence and uniqueness of the solution of problem (2). One variant of these conditions is as follows:

Theorem 4. $u \in W$, if:

1)

$$|u| < \frac{c}{r^{1/2}|\theta_1\theta_2|^{\frac{n-1}{2n-1}}}; \tag{9}$$

2)

$$\left|\frac{\partial^p u}{\partial x^p} - \chi_1 \frac{\partial^{p-1} u}{\partial x^{p-1}}\right| < \frac{c|\theta_1\theta_2|^{\frac{1}{2n-1}}}{\left(r|\theta_1\theta_2|^{\frac{2n}{2n-1}}\right)^{\lambda}}; \tag{10}$$

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3)

$$\left|\frac{\partial^p u}{\partial y^p} - \chi_2 \frac{\partial^{p-1} u}{\partial y^{p-1}}\right| < \frac{c|\theta_1\theta_2|^{\frac{1}{2n-1}}}{\left(r|\theta_1\theta_2|^{\frac{2n}{2n-1}}\right)^{\lambda}}, \tag{11}$$

where $p = 1, 2, \dots, 2n-1$; λ is arbitrary between $\frac{1}{2}$ and $\frac{1}{2} + \frac{1}{2n}$, and either

$$\chi_j = iik\left(\theta_1^{\frac{2n}{2n-1}} + \theta_2^{\frac{2n}{2n-1}}\right)^{-\frac{1}{2n}} \theta_j^{\frac{1}{2n-1}},$$

or

$$\chi_j = -ik\left(\theta_1^{\frac{2n}{2n-1}} + \theta_2^{\frac{2n}{2n-1}}\right)^{-\frac{1}{2n}} \theta_j^{\frac{1}{2n-1}}, \qquad j = 1, 2.$$

Remark 1. For $n = 1$, Theorems 3 and 4 cease to be sharp. If we were to seek the class W by this method specifically for $n = 1$, then instead of conditions (9)–(11) we would obtain the usual Sommerfeld conditions [1].

Remark 2. Denote the right-hand side of inequality (8) by g_1, and the right-hand side of inequality (9) by g_2. Then, as the right-hand side in equation (2), one may take any function satisfying the conditions

$$|g_i * f| < c|g_i|, \qquad i = 1, 2,$$

where $g_i * f$ is the convolution of these functions. These conditions are satisfied, for example, by any finite bounded function.

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Note: Figure translations are in progress. See original paper for figures.

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