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1962

SovietRxiv

Abstract

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ONE METHOD FOR CONSTRUCTING A NONSTANDARD MODEL OF THE BERNAYS-GÖDEL AXIOMATIC SET THEORY

(Presented by Academician P. S. Aleksandrov on 12 X 1961)

The initial system is the axiomatic system Σ^* of set theory (see ([1])). In this system we shall construct its model. The ordinal numbers of this model are not well ordered from the point of view of the initial theory. The general method for constructing such a model can be applied to certain special cases, for example to the construction of a model in which there are $2^{\aleph_0}$ natural numbers from the point of view of the theory in which the model is constructed.

1. In what follows let s denote an infinite set, and let I be a maximal ideal on $\mathbf{P}(s)$, if I is a set of subsets of the set s for which the following hold:

1) If $a \in I$, $b \subseteq s$, $a \subseteq b$, then $b \in I$.
2) If $a \in I$, $b \in I$, then $a \cap b \in I$.
3) If $a \cup b \in I$, then either $a \in I$ or $b \in I$.

$K(s, I)$ is the class of all functions defined on the set s. Put, for two such functions $f(x)$ and $g(x)$, $f \equiv g \pmod I$ if and only if $f(x) = g(x)$ for all x from some set $m \in I$. The relation $f \equiv g \pmod I$ is, evidently, reflexive, symmetric, and transitive.

2. Define the sets P_α for every ordinal number α by induction: $P_0 = 0$, $P_\alpha = \mathbf{P}\left(\bigcup_{\beta<\alpha} P_\beta\right)$. Evidently,

$$\bigcup_{\alpha \in O_n} P_\alpha = V$$

(their union is equal to the universal class). For every $x \in V$ there exists a first $\alpha \in O_n$ such that $x \in P_\alpha$ (denote $\alpha = \tau(x)$).

If $f \in K(s, I)$, then define

$$\mathrm{ind}_f = \inf_{m \in I} \sup_{x \in m} z(f(x)).$$

Evidently, from $f \equiv g \pmod{I}$ it follows that $\mathrm{ind}_f = \mathrm{ind}_g$.

Define the class $K'(s, I)$ in the following way:

$$f \in K'(s, I). \equiv .f \in K(s, I) \& \tau(f(x)) \leq \mathrm{ind}_f$$

for every $x \in s$. For the class $K'(s, I)$ the following hold:

1) for every $f \in K(s, I)$ there exists $g \in K'(s, I)$ such that $g \equiv f \pmod{I}$;
2) for every $f \in K'(s, I)$ the class $\overline{f}$ of all $g \in K'(s, I)$, $g \equiv f \pmod{I}$, is a set.
3. The class of all sets $\overline{f}$ (for $f \in K'(s, I)$) will be denoted by $\overline{V}$. The class $\overline{E}$ contains exactly the ordered pairs of the type $\langle \overline{f}, \overline{g} \rangle$ for which the following holds:

$$\overline{f} \in \overline{V} \ \& \ \overline{g} \in \overline{V} \ \& \ f(x) \in g(x)$$

for all x of some $m \in I$. Evidently, $\langle \overline{f}, \overline{g} \rangle \in \overline{E}$ does not depend on the choice of special functions from the sets $\overline{f}$ and $\overline{g}$.

Define for every $\overline{f} \in \overline{V}$ the class $\varphi(\overline{f})$ in the following way:

$$\overline{g} \in \varphi(\overline{f}). \equiv .\langle \overline{g}, \overline{f} \rangle \in \overline{E}.$$

4. The sets (Π^*), classes (Cls^*), and the relation $\in^*$ of the model $\Gamma(s, I)$ are defined in the following way:

$$M^*(X). \equiv .X \in \overline{V};$$

$$\mathrm{Cls}^*(X). \equiv .X \subseteq \overline{V}$$

and for every $\overline{f}$ there exists $\overline{g}$ such that

$$\varphi(\overline{g}) = X \cap \varphi(\overline{f});$$

$$x \in^* Y. \equiv .\{M^*(x) \ \& \ M^*(Y) \ \& \ \langle XY \rangle \in \overline{E}\} \ \cup \ \{M^*(x) \ \& \ \mathrm{Cls}^*(Y) \ \& \ X \in Y\}.$$

The set $\bar{f}$ is equal to the class $\varphi(\bar{f})$ (of the model $\Gamma(s, I)$). It can be proved that in this model all the axioms of the system Σ^* are satisfied.

5. The class O_n^* of the model $\Gamma(s, I)$ consists of all sets $\bar{f}$ such that, for some $m \in I$, the function $f(x)$ takes as its values only ordinal numbers of the original theory.

The natural numbers of the model are those sets $\bar{f}$ such that, for some $m \in I$, the function $f(x)$ takes as its values only natural numbers of the original theory.

In the case when I consists of all subsets of the set s containing one point, $\Gamma(s, I)$ is isomorphic to the original theory. In the remaining cases we obtain a nonstandard model of set theory. If, for example, s is a countable set, then there exist $2^{\aleph_0}$ natural numbers of the model $\Gamma(s, I)$ from the point of view of the original theory, but in this case there also exist $2^{\aleph_0}$ ordinal numbers before the first noncountable one from the point of view of the original theory.

In the case when s is uncountable and I is not equivalent to an ideal on a set of smaller cardinality, we obtain a model nonisomorphic to $\Gamma(s', I')$, where $\operatorname{card} s' < \operatorname{card} s$.

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Received
1 IX 1961

REFERENCES

1. K. Gödel, *Ann. of Math. Studies*, No. 3 (1940).

Note: Figure translations are in progress. See original paper for figures.