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THE EXACT CONSTANT IN D. JACKSON' S THEOREM ON BEST UNIFORM APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS

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Abstract

Full Text

MATHEMATICS

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THE EXACT CONSTANT IN D. JACKSON' S THEOREM ON BEST UNIFORM APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS

(Presented by Academician P. S. Novikov, 27 II 1962)

1. Let $C_{2\pi}$ be the space of continuous functions with period 2π. If the modulus of continuity $\omega(f;t)$ of a function $f \in C_{2\pi}$, for all $0 \le t \le \pi$, does not exceed some convex modulus of continuity $\omega_1(t)$, then, as was shown in the paper ([1]), for the best uniform approximation $E_n(f)$ of the function f by trigonometric polynomials of degree not exceeding n, the estimate

$$E_n(f) \le \frac{1}{2}\,\omega_1\left(\frac{\pi}{n+1}\right) \quad (n = 0, 1, 2, \ldots). \tag{1}$$

holds.

By virtue of a lemma of S. B. Stechkin (see ([2]), Lemma 4), for any modulus of continuity $\omega(t)$, $\omega(t) \not\equiv 0$, there exists a convex monotone majorant $\omega_1(t)$ such that $\omega(t) \le \omega_1(t) < 2\omega(t)$. Hence from (1) it follows at once that, whatever the function $f \in C_{2\pi}$, $f \not\equiv$ const, one has

$$E_n(f) < \omega\left(f; \frac{\pi}{n+1}\right) \quad (n = 0, 1, 2, \ldots).$$

S. B. Stechkin conjectured that the last estimate cannot be improved on the whole space $C_{2\pi}$.

We shall show that this is indeed so.

Lemma. For any fixed $n = 0, 1, 2, \ldots$, whatever the number $\varepsilon > 0$, one can specify a function $\varphi \in C_{2\pi}$, not identically constant, such that

$$E_n(\varphi) > \left(\frac{2n+1}{2n+2} - \varepsilon\right)\omega\left(\varphi; \frac{\pi}{n+1}\right). \tag{2}$$

Let $\varepsilon > 0$ be given, and we may assume $0 < \varepsilon < \frac{1}{2}$. Put

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$$h = \frac{\pi}{n+1}, \qquad x_0 = 0, \qquad x_k = kh - (n-k+1)\beta \quad (k = 1, 2, \dots, n+1),$$

where the number β satisfies the inequalities

$$0 < \beta < \frac{2\varepsilon}{(n+1)^2},$$

and therefore

$$x_0 < x_1 < x_2 < \dots < x_{n+1} = \pi.$$

Construct a continuous even function $\varphi(x)$ with period 2π, defining it on the interval $[0, \pi]$ as follows:

$$\varphi(x_k) = (-1)^{k+1} \quad (k = 1, 2, \dots, n+1),$$

$$\varphi(x) = 0, \quad \text{if } 0 \le x \le x_1 - \beta \quad \text{and} \quad x_k + \beta \le x \le x_{k+1} - \beta \quad (k = 1, 2, \dots, n+1),$$

and $\varphi(x)$ is linear on the intervals

$$[x_k - \beta, x_k] \quad (k = 1, 2, \dots, n+1)$$

and

$$[x_k, x_k + \beta] \quad (k = 1, 2, \dots, n).$$

It is easy to verify that

$$\omega\left(\varphi; \frac{\pi}{n+1}\right) = 1.$$

Let

$$T_n(x) = \frac{1}{n+1}\left(\frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx\right) = \frac{1}{n+1}\,\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}.$$

As is not difficult to compute,

$$T_n(x_0) = T_n(0) = \frac{2n+1}{2n+2}, \qquad T_n(kh) = \frac{(-1)^{k+1}}{2n+2} \quad (k = 1, 2, \dots, n+1),$$

$$|T_n(x_k) - T_n(kh)| < \frac{n}{2}(n-k+1)\beta < \varepsilon \quad (k = 1, 2, \dots, n).$$

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Consequently,

$$\varphi(x_k) - T_n(x_k) = [\varphi(x_k) - T_n(kh)] + [T_n(kh) - T_n(x_k)]$$

$$= (-1)^{k+1}\frac{2n+1}{2n+2} + \mu_k \qquad (k = 0, 1, 2, \dots, n+1),$$

where $0 \le |\mu_k| < \varepsilon < 1/2$.

Taking into account the evenness of $\varphi(x)$ and $T_n(x)$, we see that the difference $\varphi(x) - T_n(x)$ on the period $(-\pi, \pi]$ assumes, at $2n+2$ points, values with alternating signs, and these values in absolute magnitude exceed $\frac{2n+1}{2n+2} - \varepsilon$. Therefore

$$E_n(\varphi) > \frac{2n+1}{2n+2} - \varepsilon = \left(\frac{2n+1}{2n+2} - \varepsilon\right)\omega\left(\varphi; \frac{\pi}{n+1}\right),$$

and the lemma is proved.

Thus, D. Jackson' s theorem (see, for example, [3], p. 238) for the space $C_{2\pi}$ takes the following form.

Theorem. If $f(x) \in C_{2\pi}$, then

$$E_n(f) \le \omega\left(f; \frac{\pi}{n+1}\right) \qquad (n = 0, 1, 2, \dots), \tag{3}$$

where equality in relation (3) occurs only in the case when $f(x)$ is identically constant.

The absolute constant 1 on the right-hand side of inequality (3) is definitive*.

Let us note an obvious consequence.

Corollary. The limiting equality holds

$$\lim_{n\to\infty} \sup_{f \in C_{2\pi}} \frac{E_n(f)}{\omega\left(f; \frac{\pi}{n+1}\right)} = 1. \tag{4}$$

2. Let $C_{[a,b]}$ be the space of functions continuous on the interval $[a, b]$, and let $E_n(f; a, b)$ be the best uniform approximation of a function $f \in C_{[a,b]}$ on this interval by algebraic polynomials of degree $\le n$.

Then, putting

$$f\left[\frac{(b-a)\cos\theta + (a+b)}{2}\right] = \psi(\theta)$$

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and taking into account that, if $f \in C_{[a,b]}$, then $\psi \in C_{2\pi}$ and, moreover (see, for example, [4]), $E_n(f; a, b) = E_n(\psi)$, $\omega(\psi; t) \leq \omega\left(f; \frac{b-a}{2} t\right)$, with the help of (3) we arrive at the estimate

$$E_n(f; a, b) \leq \omega\left(f; \frac{b-a}{2} \frac{\pi}{n+1}\right) \qquad (n = 0, 1, 2, ...),$$

valid for every function f of the space $C_{[a,b]}$.

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[3] V. L. Goncharov, *Theory of Interpolation and Approximation of Functions*, Moscow, 1954.
[4] I. P. Natanson, *Constructive Theory of Functions*, Moscow–Leningrad, 1949.

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* That is, inequality (3) ceases to be true simultaneously for all $f \in C_{2\pi}$ and $n = 0, 1, 2, ...$, if its right-hand side is multiplied by $1 - \varepsilon$, where ε is a positive number independent of both f and n.

Note: Figure translations are in progress. See original paper for figures.

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