B. M. Urazbaev

1962

SovietRxiv

Abstract

Full Text

B. M. Urazbaev

On the Growth of the Number of Special Absolutely Abelian Fields

(Presented by Academician I. M. Vinogradov, 31 VII 1961)

The note considers the question of the distribution of certain special absolutely abelian fields as their discriminant grows. Absolutely abelian fields whose discriminants satisfy given requirements are called special fields. Among special fields are included, for example, cyclic fields of any prime degree $l \geqslant 2$ with a fixed number of critical prime divisors; cyclic fields of degree l^h, $h \geqslant 1$, in which all critical prime divisors are completely critical ((1, p. 64); absolutely abelian fields whose components contain a prescribed number of critical prime divisors, etc.

There are, obviously, infinitely many special fields of each given type, and the laws of their distribution can be represented in the form of certain asymptotic formulas. The construction of asymptotic formulas rests essentially on the theory of Dirichlet L-functions (2) and on the results of N. G. Chudakov concerning bounds for the zeros of Dirichlet L-functions in the critical strip (3).

Lemma 1. *The number of absolutely abelian fields representing the composite of two cyclic fields of prime degree l, of which at least one is a field with a single critical prime divisor, is equal to*

$$k(l-1)^{k-2},$$

where k is the number of prime divisors of the discriminant of the composite, and l is a noncritical prime.

Lemma 2. *The number of absolutely abelian fields constituting the composite of two cyclic fields of degree l, of which at least one field has a prescribed number m of critical prime divisors, is equal to*

$$C_k^m (l-1)^{k-2}.$$

For the proof of the lemmas see (5).

Theorem 1. *The law of growth of the number $N_1(x)$ of special absolutely abelian fields of the kind indicated in Lemma 1, whose discriminants are $\leqslant x^{l(l-1)}$, is represented by the asymptotic formula*

$$N_1(x) = \lambda_0 x(\ln\ln x + C) + O(xe^{-\theta(\ln x)^{\mu}}),$$

where $\lambda_0 > 0$ *is a structural constant;* $\mu = \dfrac{1}{2} + \dfrac{1}{42} - \varepsilon$ *is Chudakov' s constant* (^{3})*;* $C = 0.577\ldots$ *is Euler' s constant;* $\theta > 0$ *is a quantity depending on* l *and* ε*; and* $\varepsilon > 0$ *is arbitrary.*

For the proof see (^{5,6}).

In the present note we shall prove

Theorem 2. *The law of growth of the number* $N_2(x)$ *of special absolutely abelian fields of the kind indicated in Lemma 2, whose discriminants are* $\leqslant x^{l(l-1)}$*, is represented by the asymptotic formula*

$$N_2(x) = xf(\ln\ln x) + O(xe^{-\theta(\ln x)^{\mu}}), \tag{1}$$

where $f(\ln\ln x)$ *is a polynomial of degree* m *in* $\ln\ln x$*, whose coefficients depend only on* l *and* m*;* θ *and* μ *have the values indicated in Theorem 1.*

Proof. Consider the Dirichlet series

$$L(s, \tau; \chi) = \sum_{n=1}^{\infty} \frac{\tau^{\nu(n)}\chi(n)}{n^s}, \qquad s = \sigma + it, \tag{2}$$

where $l \geq 2$ is a fixed prime number; χ is a Dirichlet character $(\operatorname{mod} l)$; $\nu(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the canonical factorization of n; τ is a variable parameter, $0 < \tau \leq 1$. The series (2) converges absolutely and uniformly in the half-plane $\sigma \geq \sigma_0 > 1$, and the function $L(s, \tau; \chi)$ represented by the series (2) is a regular analytic function in the half-plane $\sigma > 1$. For $\sigma > 1$ the function $L(s, \tau; \chi)$ expands into the infinite product

$$L(s, \tau; \chi) = \prod_p \left(1 - \frac{\tau\chi(p)}{p^s}\right)^{-1}, \tag{3}$$

where p runs through all prime numbers.

The function $L(s, \tau; \chi)$ is connected with the Dirichlet L-function $L(s; \chi)$ by a relation of the form $(\sigma > 1)$

$$L(s, \tau; \chi) = L^{\tau}(s; \chi)\psi(s, \tau; \chi), \tag{4}$$

where $\psi(s, \tau; \chi)$ is a regular function, uniformly bounded in the half-plane $\sigma > 1/2$. Equality (4) shows that the zeros of $L(s; \chi_0)$ are critical points of $L(s, \tau; \chi_0)$, if χ_0 is the principal character. If $\chi \neq \chi_0$, then the function $L(s, \tau; \chi_0)$ is regular in a neighborhood of the point $s = 1$ and at this point itself has a finite value different from zero.

Consider the function $f(s, \tau)$, defined by the infinite product

$$f(s,\tau) = \prod_{p\equiv 1\,(l)} \left(1 + \frac{\tau(l-1)}{p^s}\right).$$

We have ($\sigma > 1$)

$$f(s,\tau) = \sum_{n=1}^{\infty} \frac{l(n,\tau)}{n^s}, \tag{5}$$

where

$$l(n,\tau) = \begin{cases} \tau^k(l-1)^k, & \text{if } n = p_1p_2\cdots p_k,\ p_i \equiv 1\,(l) \text{ are distinct prime numbers;} \\ 0, & \text{if } n \neq p_1p_2\cdots p_k. \end{cases}$$

The series (5) converges absolutely and uniformly for $\sigma \geq \sigma_0 > 1$, and the function $f(s,\tau)$ is regular there. From (2)–(4) we find ($\sigma > 1$)

$$f(s,\tau) = \left(\prod_{\chi} L(s;\chi)\right)^{\tau} \varphi(s,\tau;\chi), \tag{6}$$

where $\varphi(s,\tau;\chi)$ is a regular function, uniformly bounded in the half-plane $\sigma > 1/2$ for all values of τ, $0 < \tau \leq 1$, and all χ.

According to the results of N. G. Chudakov (^{3}), the Dirichlet L-functions corresponding to the fixed prime modulus l have a zero-free region

$$\sigma \geq 1 - \eta(\ln|t| + 3)^{-\alpha}, \qquad -\infty < t < +\infty, \tag{7}$$

where $\alpha = \frac{10}{11} + \varepsilon$; $\varepsilon > 0$ is arbitrary; $\eta > 0$ is a constant depending on l and ε, $0 < \eta < 1$.

The function $f(s,\tau)$ is regular everywhere in the region (7), with the exception of the single point $s = 1$, which is its branch point. Let Γ_1 be the curvilinear quadrilateral bounded by the curve (7) and the straight lines $\sigma = 2$, $t = \pm T$, $T \geq T_0 > 1$; let Γ_2 be the contour consisting of the circle C_ρ, sufficiently ⋯

exactly a small radius ρ, described about the point $s = 1$, and the segment of the cut $(-\infty, +1)$ from the point a to $1-\rho$, where $a = 1 - \eta(\ln 3^{-\alpha})$.

Consider the integral

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f(s,\tau)}{s(s+1)} x^s\, ds,$$

where $\Gamma = \Gamma_1 + \Gamma_2$. Using Cauchy' s theorem, we find (^{6})

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{f(s,\tau)}{s(s+1)} x^s\,ds =$$

$$= A + B + \frac{1}{2\pi i}\int_{\beta^-} \frac{f(s,\tau)}{s(s+1)} x^s\,ds + \frac{1}{2\pi i}\int_{\delta^-} \text{idem} + \frac{1}{2\pi i}\int_{\omega^-} \text{idem}, \tag{8}$$

where β, δ are the bases of the curvilinear quadrilateral Γ_1; ω is the curvilinear side of the contour Γ_1, bounded by the curve (7);

$$A = \frac{\lambda x}{\pi} \sin \pi\tau \cdot \ln^{\tau-1} x\, \Gamma(1-\tau), \qquad B = \frac{\lambda x}{\pi} \sin \pi\tau \cdot \ln^{\tau-1} x \int_{(1-a)\ln x}^{\infty} z^{-\tau} e^{-z}\,dz.$$

The function $f(s,\tau)$ is uniformly continuous and has continuous derivatives with respect to τ, $0 < \tau \le 1$. Denote by $f^{(m)}(s,\tau)$ the partial derivative of $f(s,\tau)$ with respect to τ. From (6) we find

$$f^{(m)}(s,\tau) = \left(\prod_{\chi} L(s;\chi)\right)^{\tau} \Phi(s,\tau;\chi), \tag{9}$$

where $\Phi(s,\tau;\chi)$ is a regular function, uniformly bounded in the domain (7).

Starting from the estimate for $f(s,\tau)$ (^{6}), it is not difficult to obtain the estimate

$$|f^{(m)}(s,\tau)| = O\left(|t|^{\frac{1-\sigma}{2}(l-1)\tau+\varepsilon}\right) \qquad (|t| \ge t_0 > 1) \tag{10}$$

uniformly in the domain $1/2 + \varepsilon < \sigma < 1$ for all τ, $0 < \tau \le 1$; $\varepsilon > 0$ is arbitrary. In turn, from (10) it follows that

$$\left|\frac{f^{(m)}(s,\tau)}{s(s+1)}\right| = O\left(|t|^{1-\eta-\varepsilon}\right) \tag{11}$$

uniformly for $\sigma > 1 - \dfrac{2(1-\eta)}{l-1} + \varepsilon$. Using estimate (11), we find

$$\left|\frac{1}{2\pi i}\int_{\omega^-} \frac{f^{(m)}(s,\tau)}{s(s+1)} x^{s-1}\,ds\right| < M e^{-\theta(\ln x)^{\mu}}, \tag{12}$$

where M is a positive number $< \infty$. It also follows from estimate (11) that, as $T \to \infty$, the integrals on the right in (8), taken over the contours β^- and δ^-, tend to zero. Passing to the limit as $T \to \infty$, we find from (8) and (12)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{f^{(m)}(s,\tau)}{s(s+1)}x^s\,ds = A_m x + (B_{m-1}T_0 + B_{m-2}T_1 + \dots$$
$$\dots + B_0 T_{m-1})x + (1-\tau)H(x,\tau) + O\left(xe^{-\theta \ln x^{\mu}}\right), \tag{13}$$

where A_n, B_n are polynomials of degree n in $\ln\ln x$, whose coefficients depend only on l and m;

$$T_n = \int_{(1-a)\ln x}^{\infty} z^{-1}e^{-z}\ln z\,dz, \qquad (1-a)\ln x \geq 1 \quad (n = 0, 1, 2, \dots).$$

Introduce the functions

$$\varphi(x,\tau) = \sum_{n\leq x} I^{(m)}(n,\tau), \qquad \varphi_1(x,\tau) = \int_1^x \varphi(u,\tau)\,du,$$

where

$$I^{(m)}(n,\tau) = k(k-1)\cdots(k-m+1)\tau^{k-m}(l-1)^k.$$

We have (4)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{f^{(m)}(s,\tau)}{s(s+1)}x^s\,ds = \sum_{n\leq x}\left(1-\frac{n}{x}\right)I^{(m)}(n,\tau) = \frac{1}{x}\varphi_1(x,\tau). \tag{14}$$

We now pass in (13) to the limit as $\tau \to 1$. Then, taking (14) into account, we shall have

$$\frac{1}{x}\varphi_1(x) = O\left(\frac{x^a(\ln\ln x)^{m-1}}{\ln x}\right) + O\left(xe^{-\theta(\ln x)^{\mu}}\right), \tag{15}$$

since the quantity $x(B_{m-1}T_0 + B_{m-2}T_1 + \dots + B_0T_{m-1})$ has order

$$O\left(\frac{x^a(\ln\ln x)^{m-1}}{\ln x}\right).$$

Here

$$\varphi_1(x) = \lim_{\tau\to 1}\varphi_1(x,\tau),$$

$$\varphi(x) = \lim_{\tau\to 1}\varphi(x,\tau) = \sum_{p_1p_2\cdots p_k\leq x} k(k-1)\cdots(k-m+1)(l-1)^k,$$

$$\varphi_1(x) = \int_1^x \varphi(u)\,du.$$

Now, proceeding from (15), it is no longer difficult to obtain in the usual way (4) the desired asymptotic formula (1). This completes the proof of Theorem 2.

Received
19 VII 1961

References

[1] N. G. Chebotarev, *Foundations of Galois Theory*, Part II, 1937.
[2] N. G. Chudakov, *Introduction to the Theory of Dirichlet L-Functions*, 1947.
[3] N. G. Chudakov, *Matem. sborn.*, 1(43), 591 (1936).
[4] A. E. Ingham, *The Distribution of Prime Numbers*, 1936.
[5] B. M. Urazbaev, Proceedings of the Third All-Union Mathematical Congress, 1, 1956, p. 35.
[6] B. M. Urazbaev, *Izv. AN Kaz. SSR, Ser. Math. and Mech.*, 8(12), 70 (1959).

Note: Figure translations are in progress. See original paper for figures.