MATHEMATICS

B. M. BREDIKHIN

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

B. M. BREDIKHIN

BINARY ADDITIVE PROBLEMS WITH PRIME NUMBERS

(Presented by Academician P. S. Novikov on 25 IX 1961)

The dispersion method, developed by Yu. V. Linnik in a series of recent works ([1–5]), was considerably strengthened and simplified by him through the introduction of the notion of covariance and coherent numbers ([6,7]). In connection with this, the range of application of the dispersion method expanded; in particular, it became possible to treat by the dispersion method certain binary problems with prime numbers of an "indefinite type" :

$$p - \varphi(x, y) = a, \tag{1}$$

where a is a given integer different from zero, $\varphi(x, y)$ is a given binary quadratic form, and p runs through the prime numbers.

In this note two theorems are considered which furnish asymptotics for the number of solutions of equation (1) in the cases where $\varphi(x, y) = xy$ and $\varphi(x, y) = x^2 + y^2$.

First consider the equation

$$p - xy = a, \tag{2}$$

where x and y independently run through the natural numbers under the condition $xy \leq n$. Let $Q(n)$ be the number of solutions of equation (2).

Theorem 1. As $n \to \infty$,

$$Q(n) = \frac{315\zeta(3)}{2\pi^4} \prod_{p/a} \frac{(p-1)^2}{p^2 - p + 1} n + R(n), \tag{3}$$

where

$$R(n) = O(n(\ln n)^{-0.999}).$$

Since

$$Q(n) = \sum_{0<p-a\le n} \tau(p-a),$$

where $\tau(m)$ is the number of divisors of m, Theorem 1 evidently gives a complete solution of the divisor problem for shifted primes, posed by E. C. Titchmarsh in 1930 ([8]), which consisted in finding the asymptotics of the expression

$$\sum_{p\le n} \tau(p-a)$$

as $n \to \infty$ and for given a (in the monograph of Yu. V. Linnik ([7]), in Chapter VIII, the solution of Titchmarsh' s problem is considered for $a = 1$).

Now consider the equation

$$p - x^2 - y^2 = a, \tag{4}$$

where x and y independently run through the integers under the condition $x^2 + y^2 \le n$. Let $S(n)$ be the number of solutions of equation (4).

Theorem 2. As $n \to \infty$,

$$S(n) = \pi \frac{n}{\ln n} \prod_{p>2} \left(1 + \frac{\chi_4(p)}{p(p-1)}\right) \prod_{\substack{p/a \\ p\equiv 1 \pmod 4}} \frac{(p-1)^2}{p^2-p+1} \prod_{\substack{p/a \\ p\equiv 3 \pmod 4}} \frac{p-1}{p^2-p+1} + R(n), \tag{5}$$

where $\chi_4(m)$ is the nonprincipal character $(\mathrm{mod}\, 4)$ and

$$R(n) = O(n(\ln n)^{-1.028}).$$

Theorem 2 is related to the Hardy—Littlewood problem. Let us note an interesting corollary of this theorem.

Corollary of Theorem 2. There exist infinitely many primes of the form

$$x^2 + y^2 + a.$$

Before the creation of the dispersion method, Theorem 2, as was indicated by C. Hooley ([9]), could have been proved only with the use of the extended Riemann hypothesis. The same can also be said about Theorem 1.

The dispersion method, in combination with coherent numbers and the results of C. Hooley, makes it possible to give unconditional proofs of Theorems 1 and 2.

Here we shall confine ourselves to presenting the scheme of the proof of Theorem 1.

Along with equation (2), consider the equations

$$p - xy = a_i, \tag{6}$$

where $a_i = aq_i$, $q_i \in (\frac{1}{2}n^{1-\varepsilon}, n^{1-\varepsilon})$; $\varepsilon > 0$ is sufficiently small; q_i is almost-prime, i.e. contains no prime divisors; $p \leqslant P = n^{1/(\ln\ln n)^2}$.

Let $Q_i(n)$ be the number of solutions of equation (6) under the condition $xy \leqslant n$; $i = 1, 2, \dots, m$, where m is the number of almost-prime numbers in the interval $(\frac{1}{2}n^{1-\varepsilon}, n^{1-\varepsilon})$.

Comparing $Q_i(n)$ with $Q(n)$, we find, using the basic theorems of the dispersion method of Yu. V. Linnik ([7]), that

$$Q_i(n) = Q(n) + O(n(\ln n)^{-0.999}). \tag{7}$$

In doing this it is necessary only to introduce a slight modification into the proofs of the theorems of the dispersion method, adapting them to the system of coherent numbers a_i chosen by us.

On the other hand, elementary transformations of $Q_i(n)$ and the application of upper estimates for the number of primes in arithmetic progressions ([10]) yield the equality

$$Q_i(n) = 2 \sum_{x \leqslant \sqrt{n}} \sum_{\substack{p - xy = a_i \\ xy < n}} 1 + O\left(\frac{n}{\ln n}(\ln\ln n)^2\right). \tag{8}$$

From (7) and (8), by summing over all values $i = 1, 2, \dots, m$, we derive the equality

$$Q(n) = \frac{2}{m}\widetilde{Q}(n) + O(n(\ln n)^{-0.999}), \tag{9}$$

where

$$\widetilde{Q}(n) = \sum_{i=1}^{m} \sum_{x \leqslant \sqrt{n}} \sum_{\substack{p - xy = a_i \\ xy < n}} 1.$$

It remains to find an asymptotic formula for $\widetilde{Q}(n)$. This is already a ternary problem, which is solved directly by means of C. Hooley' s lemma ([9]) on the distribution of almost-prime numbers in arithmetic progressions.

Transforming $\widetilde{Q}(n)$, we obtain the equality

$$\widetilde{Q}(n) = \sum_{\substack{x \leqslant \sqrt{n} \\ (x,a)=1}} \sum_{x+an^{1-\varepsilon}<p<n+x/2an^{1-\varepsilon}} \sum_{\substack{q_i \equiv pa' \pmod x \\ q_i \in (\frac{1}{2}n^{1-\varepsilon},\, n^{1-\varepsilon})}} 1 + O(mn^{1-\varepsilon}), \qquad (10)$$

where $aa' \equiv 1 \pmod x$.

Estimating the inner sum in equality (10) by means of Lemma C. Hooley, we derive the equality

$$\widetilde{Q}(n) = m \sum_{\substack{x \leq \sqrt{n} \\ (x,a)=1}} \frac{1}{\varphi(x)} \frac{n}{\ln n} + O\Big(m \frac{n}{\ln n}\Big). \qquad (11)$$

After carrying out the summation on the right-hand side of (11), from equalities (9) and (11) we obtain (3), which completes the proof of Theorem 1.

The proof of Theorem 2 is carried out according to the same scheme, with certain complications.

I consider it a pleasant duty to express my deep gratitude to Yu. V. Linnik for valuable advice and attention to my work.

Kuibyshev Pedagogical Institute
named after V. V. Kuibyshev

Received
21 IX 1961

References

1. Yu. V. Linnik, Tr. Mat. Inst. Akad. Nauk Vengrii, **4**, No. 3–4 (1959).

2. Yu. V. Linnik, Matem. sborn., **51** (93), No. 2 (1960).

3. Yu. V. Linnik, Matem. sborn., **52** (94), No. 2 (1960).

4. Yu. V. Linnik, Matem. sborn., **53** (95), No. 1 (1961).

5. Yu. V. Linnik, Izv. AN SSSR, ser. matem., **24**, No. 5 (1960).

6. Yu. V. Linnik, DAN, **137**, No. 6 (1961).

7. Yu. V. Linnik, *The Dispersion Method in Binary Additive Problems*, Leningrad State University Press, 1961.

8. E. C. Titchmarsh, Rend. Circ. Mat. Palermo, **54** (1930).

9. C. Hooley, Acta Math., **97**, 189 (1957).

10. I. V. Chulakovskii, DAN, **63**, No. 5 (1948).

Note: Figure translations are in progress. See original paper for figures.