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# MATHEMATICS

V. G. VIZING

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**Abstract**

**Full Text**

MATHEMATICS

V. G. VIZING

## EQUIVALENT FLOWS IN A TRANSPORT NETWORK

(Presented by Academician S. L. Sobolev, 5 II 1962)

A. Let there be given a finite transport network  $G(X, U)$  with input  $x_0$ , output  $z$ , and capacities  $c(u) \geq 0$ , where  $u \in U$  (see (1), Ch. VIII). Two flows  $\varphi$  and  $\varphi'$  in the network  $G$  are called **equivalent** if  $\varphi_z = \varphi'_z$ . The **distance** between the flows  $\varphi$  and  $\varphi'$  will be the quantity

$$\rho(\varphi, \varphi') = \sum_{u \in U} |\varphi(u) - \varphi'(u)|.$$

A  $(\varphi, \varphi')$ -**cycle** will mean an elementary cycle composed of arcs  $u \in U$  such that on arcs of one direction  $\varphi(u) < \varphi'(u)$ , and on arcs of the opposite direction  $\varphi(u) > \varphi'(u)$ . A  $\varphi$ -**cycle** will mean an elementary cycle composed of arcs  $u \in U$  such that on arcs of one direction  $\varphi(u) < c(u)$ , and on arcs of the opposite direction  $\varphi(u) > 0$ .

**Theorem 1\***. Let  $\varphi$  be a flow in the network  $G(X, U)$ . For the existence of an equivalent flow  $\varphi'$  such that  $\rho(\varphi, \varphi') > 0$ , it is necessary and sufficient that a  $\varphi$ -cycle exist.

**Sufficiency.** Suppose that a  $\varphi$ -cycle exists. Put:  $\varphi'(u) = \varphi(u) + 1$  on those arcs of the cycle on which  $\varphi(u) < c(u)$ ;  $\varphi'(u) = \varphi(u) - 1$  on those arcs of the cycle on which  $\varphi(u) > 0$ ;  $\varphi'(u) = \varphi(u)$  on the remaining arcs of the transport network  $G$ .

We have  $\varphi_z = \varphi'_z$  and  $\rho(\varphi, \varphi') > 0$ .

**Necessity.** Suppose that there exist two equivalent flows  $\varphi$  and  $\varphi'$  in the network  $G$ , with  $\rho(\varphi, \varphi') > 0$ . Since a  $(\varphi, \varphi')$ -cycle is obviously a  $\varphi$ -cycle, it is enough to prove the existence of a  $(\varphi, \varphi')$ -cycle.

If on  $u_1 \in U_z^-$  (or  $u_1 \in U_{x_0}^+$ )  $\varphi(u_1) > \varphi'(u_1)$ , then, owing to the equivalence of the flows  $\varphi$  and  $\varphi'$ , there exists an arc  $u_2 \in U_z^-$  ( $u_2 \in U_{x_0}^+$ ) on which  $\varphi(u_2) < \varphi'(u_2)$ . If, however, for  $x \neq x_0$ ,  $x \neq z$ , there exists an arc  $u \in U_x^-$  on which  $\varphi(u) > \varphi'(u)$  ( $\varphi(u) < \varphi'(u)$ ), then, since  $\varphi$  and  $\varphi'$  are flows and the transport network has no loops, there exists a vertex  $y$ , distinct from  $x$ , such that:

either  $\varphi(x, y) > \varphi'(x, y)$  ( $\varphi(x, y) < \varphi'(x, y)$ );

or  $\varphi(y, x) < \varphi'(y, x)$  ( $\varphi(y, x) > \varphi'(y, x)$ ).

Thus there exists an infinite  $(\varphi, \varphi')$ -chain; and since the transport network is finite, the chain closes, and the first closure gives us the required  $(\varphi, \varphi')$ -cycle.

Theorem 1 is proved.

B. From the proof of Theorem 1 it follows that if  $\varphi'_z = \varphi_z$  and  $\rho(\varphi, \varphi') > 0$ , then the flow  $\varphi'$  can be obtained from  $\varphi$  in the following way.

Find a  $(\varphi, \varphi')$ -cycle. Put:

$$\begin{aligned} \varphi''(u) &= \varphi(u) + 1 && \text{on those arcs of the cycle on which } \varphi(u) < \varphi'(u); \\ \varphi''(u) &= \varphi(u) - 1 && \text{on those arcs of the cycle on which } \varphi(u) > \varphi'(u); \\ \varphi''(u) &= \varphi(u) && \text{on the remaining arcs of the transport network.} \end{aligned}$$

\* **Note added in proof.** After the paper had been submitted for publication, it became known to the author that, in a somewhat more general form, a theorem of analogous content was proved in the work of Hwang Tua (2).

We have:  $\varphi''_z = \varphi_z - \varphi'_z$  and  $\rho(\varphi'', \varphi') < \rho(\varphi, \varphi')$ .

If  $\rho(\varphi'', \varphi') > 0$ , then we find a  $(\varphi'', \varphi')$ -cycle, construct a flow  $\varphi'''$ , etc. Since  $\rho(\varphi, \varphi')$  is a nonnegative integer, after a finite number of such steps we obtain a flow  $\tilde{\varphi}$  such that  $\rho(\tilde{\varphi}, \varphi') = 0$ .

B. Here we shall generalize to  $p$ -graphs Theorem 2 from Chapter IX <sup>(1)</sup>.

Let there be a  $p$ -graph  $G = (X, U)$  with vertices  $\{x_i\}$ ,  $i = 1, 2, \dots, n$ , and semidegrees  $(r_i, s_i)$ ,  $i = 1, 2, \dots, n$  ( $r_i$  is the out-semidegree,  $s_i$  the in-semidegree of the vertex  $x_i$ ).

Denote by  $\alpha_G(x, y)$  the number of arcs in  $U$  going from  $x$  to  $y$ . Let  $x, y, x', y'$  be vertices of  $G$ , with  $x \neq x'$ ,  $y \neq y'$ ;  $\alpha_G(x, y) > 0$ ;  $\alpha_G(x', y') > 0$ ;  $\alpha_G(x, y') < p$ ;  $\alpha_G(x', y) < p$ .

We shall say that a  $p$ -graph  $G' = (X, V)$  is obtained from  $G = (X, U)$  by one transfer if it differs from  $G$  only in that

$$\alpha_{G'}(x, y) = \alpha_G(x, y) - 1; \quad \alpha_{G'}(x', y') = \alpha_G(x', y') - 1;$$

$$\alpha_{G'}(x, y') = \alpha_G(x, y') + 1, \quad \alpha_{G'}(x', y) = \alpha_G(x', y) + 1.$$

It is clear that a transfer carries a  $p$ -graph into a  $p$ -graph with the same semidegrees.

**Theorem 2.** *Let  $G = (X, U)$  and  $H = (X, V)$  be two  $p$ -graphs with respectively equal semidegrees. Then  $H$  can be obtained from  $G$  by a finite number of transfers.*

Construct a transportation network  $T$  with input  $x_0$ , output  $z$ , and vertices

$$x_1, x_2, \dots, x_n, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n.$$

Join the vertices  $x_i$  and  $\bar{x}_j$  by an arc of capacity  $c(x_i, \bar{x}_j) = p$ , the vertices  $x_0$  and  $x_i$  by an arc of capacity  $c(x_0, x_i) = r_i$ , and the vertices  $\bar{x}_j$  and  $z$  by an arc of capacity  $c(\bar{x}_j, z) = s_j$ . Then the  $p$ -graph  $G$  determines a flow  $\varphi$  in the network  $T$ , saturating the input and output arcs, with  $\varphi(x_i, \bar{x}_j) = \alpha_G(x_i, x_j)$ ; the  $p$ -graph  $H$  correspondingly determines a flow  $\varphi'$ , of the same value as  $\varphi$ . The theorem will be proved if we show that, by a finite number of transformations of the flow equivalent to transfers in the  $p$ -graph, one can obtain from  $\varphi$  a flow  $\tilde{\varphi}$  of the same value as  $\varphi$  and such that  $\rho(\tilde{\varphi}, \varphi') < \rho(\varphi, \varphi')$ .

Obviously, all  $(\varphi, \varphi')$ -cycles will consist of arcs of the form  $(x_i, \bar{x}_j)$ . Consider an alternating  $(\varphi, \varphi')$ -cycle, which, for simplicity, we denote by

$$\mu = [(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_3, y_2), \dots, (x_m, y_m), (x_1, y_m)].$$

The vertices  $x_i$  and  $x_{i+1}$  for  $i < m$ , and also  $x_m$  and  $x_1$ , will be called adjacent. We shall assume that

$$\varphi(x_i, y_i) > \varphi'(x_i, y_i), \quad i = 1, 2, \dots, m;$$

$$\varphi(x_{i+1}, y_i) < \varphi'(x_{i+1}, y_i), \quad i = 1, 2, \dots, m-1; \quad \varphi(x_1, y_m) < \varphi'(x_1, y_m).$$

For each vertex  $x_j$ , denote by  $\Gamma_{\varphi, \varphi'}(x_j)$  the set of vertices among  $\{y_i\}$ ,  $i = 1, 2, \dots, m$ , which are the ends of arcs going out of  $x_j$ , on which either  $\varphi > \varphi'$  or  $\varphi = p$ . Obviously,  $y_j \in \Gamma_{\varphi, \varphi'}(x_j)$ .

There exist two adjacent vertices  $x$  and  $x'$  such that

$$\Gamma_{\varphi, \varphi'}(x') \not\subset \Gamma_{\varphi, \varphi'}(x) \quad \text{and} \quad \Gamma_{\varphi, \varphi'}(x) \not\subset \Gamma_{\varphi, \varphi'}(x').$$

Indeed, otherwise, since  $y_i \notin \Gamma_{\varphi, \varphi'}(x_{i+1})$  for  $i < m$  and  $y_m \notin \Gamma_{\varphi, \varphi'}(x_1)$ , we would obtain

$$\Gamma_{\varphi, \varphi'}(x_1) \supset \Gamma_{\varphi, \varphi'}(x_2) \supset \dots \supset \Gamma_{\varphi, \varphi'}(x_m) \supset \Gamma_{\varphi, \varphi'}(x_1),$$

which contradicts the notion of strict inclusion.

Without loss of generality, suppose that

$$\Gamma_{\varphi, \varphi'}(x_1) \not\subset \Gamma_{\varphi, \varphi'}(x_2) \quad \text{and} \quad \Gamma_{\varphi, \varphi'}(x_2) \not\subset \Gamma_{\varphi, \varphi'}(x_1).$$

We have:  $y_1 \in \Gamma_{\varphi, \varphi'}(x_1)$ ;  $y_1 \notin \Gamma_{\varphi, \varphi'}(x_2)$ , and for some  $k \geq 2$ ,

$$y_k \in \Gamma_{\varphi, \varphi'}(x_2), \quad y_k \notin \Gamma_{\varphi, \varphi'}(x_1).$$

Put:

$$\varphi_1(x_1, y_1) = \varphi(x_1, y_1) - 1, \quad \varphi_1(x_2, y_k) = \varphi(x_2, y_k) - 1;$$

$$\varphi_1(x_1, y_k) = \varphi(x_1, y_k) + 1, \quad \varphi_1(x_2, y_1) = \varphi(x_2, y_1) + 1,$$

$$\varphi_1(u) = \varphi(u) \quad \text{on the remaining arcs of the network } T.$$

We obtain a flow  $\varphi_1$  of the same value as  $\varphi$ . Such a transformation of the flow is equivalent to a transfer in the  $p$ -graph. Obviously,

$$\rho(\varphi_1, \varphi') \leq \rho(\varphi, \varphi').$$

If  $\rho(\varphi_1, \varphi') < \rho(\varphi, \varphi')$ , then the required assertion is proved. If, however,  $\rho(\varphi_1, \varphi') = \rho(\varphi, \varphi')$ , then, as before, we consider the  $(\varphi_1, \varphi')$ -cycle

$$\mu_1 = [(x_1, y_k), (x_{k+1}, y_k), (x_{k+1}, y_{k+1}), \dots, (x_m, y_m), (x_1, y_m)],$$

which already consists of a smaller number of arcs, and in the same way perform a transfer. After a certain number of steps we either reduce  $\rho(\varphi, \varphi')$ , or obtain a flow  $\varphi''$  such that  $\rho(\varphi'', \varphi') = \rho(\varphi, \varphi')$  and the  $(\varphi'', \varphi')$ -cycle consists of four arcs. But then by the next transfer we transform the flow  $\varphi''$  into a flow  $\tilde{\varphi}$  such that

$$\rho(\tilde{\varphi}, \varphi') = \rho(\varphi'', \varphi') - 4 = \rho(\varphi, \varphi') - 4 < \rho(\varphi, \varphi').$$

Theorem 2 is proved.

G. Theorem 3, given below, solves the problem posed in (1) (Appendix IV) concerning the maximum matchings of a simple graph. Let us recall several definitions.

A **matching** of a graph  $(X, Y, \Gamma)$  is a set  $W$  of its arcs in which no two arcs are adjacent. The arcs of the matching  $W$  will be called **strong**, and the remaining arcs of the graph  $(X, Y, \Gamma)$  **weak**; the endpoints of strong arcs will be called **saturated vertices**.

Let  $W$  be a maximum matching, and let  $\mu$  be an elementary cycle of even length formed by alternating strong and weak edges. Then a new maximum matching may be defined as follows:

$$W' = (W \setminus \mu) \cup (\mu \setminus W).$$

We shall call such an operation a **cyclic transfer**.

**Theorem 3.** *Let  $W$  and  $W'$  be two maximum matchings of a simple graph  $(X, Y, \Gamma)$ . In order that  $W'$  can be obtained from  $W$  by a finite number of cyclic transfers, it is necessary and sufficient that  $W$  and  $W'$  have the same saturated vertices.*

**Necessity** follows from the fact that a cyclic transfer neither destroys old saturated vertices nor creates new saturated vertices.

**Sufficiency.** Consider the transportation network  $G$  defined by the graph  $(X, Y, \Gamma)$ , with an input  $x_0$  connected to each vertex  $x_i \in X$ , and an output  $z$  connected to each vertex  $y_i \in Y$ ; all arcs of the transportation network have capacity  $c(u) = 1$ .

If  $\varphi$  is some maximum flow in  $G$ , then the arcs of the graph  $(X, Y, \Gamma)$  on which  $\varphi = 1$  form a maximum matching. Conversely, to every maximum matching there corresponds some maximum flow in  $G$ . Now the sufficiency follows immediately from A and B.

Theorem 3 is proved.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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