M. I. Freidlin

1962

SovietRxiv

Abstract

Full Text

M. I. Freidlin

A Mixed Boundary-Value Problem for Elliptic Differential Equations of the Second Order with a Small Parameter

(Presented by Academician A. N. Kolmogorov, 8 XII 1961)

Let, in n-dimensional Euclidean space R^n, a domain D with boundary Γ be given. Suppose that the direction cosines of the normal $n(x)$ to Γ belong to the class $C^{(3)}$. Consider the operator L^ε in the domain D

$$L^\varepsilon = \frac{\varepsilon^2}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x^i} = \varepsilon^2 L_1 + L_2. \tag{1}$$

Suppose that the coefficients of this operator are three times continuously differentiable up to the boundary and that the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j$$

is nondegenerate in $D \cup \Gamma$.

Let Γ_2 be a subset of Γ, open relative to Γ, and let $\Gamma_1 = \Gamma \setminus \Gamma_2$. Let $\psi(x)$ be a continuous function on Γ_1; let $l(x)$, $x \in \Gamma$, be a vector field of class C^3, and let $\cos(n(x), l(x)) \geq \theta > 0$. In the present note we study the limiting behavior, as $\varepsilon \to 0$, of the solution of the following problem:

$$L^\varepsilon u^\varepsilon(x) = 0 \quad \text{for } x \in D; \qquad u^\varepsilon(x)\big|_{x\in\Gamma_1} = \psi(x); \qquad \frac{\partial u^\varepsilon}{\partial l}\bigg|_{x\in\Gamma_2} = 0. \tag{2}$$

To investigate the asymptotics of the solution of problem (2), we shall construct a Markov process X^ε (see ([1])) such that the solution of problem (2) is the mathematical expectation of a certain functional of the trajectories of the process X^ε, and we shall study the behavior of the trajectories of this process as $\varepsilon \to 0$. We outline the plan for constructing the process X^ε.

Consider another copy D' of the set D. Put $S = D \cup D' \cup \Gamma$. By x and x' we denote the "identical" points of the sets D and D', respectively. Define a mapping φ of the manifold S onto itself: $\varphi(x) = x'$ for $x \in D$; $\varphi(x') = x$ for $x' \in D'$, and $\varphi(y) = y$ for $y \in \Gamma$. By a neighborhood of a point $y \in \Gamma \subset S$ we shall mean a set $U \cup \varphi(U)$, where U is a set, open relative to $D \cup \Gamma$, containing the point y.

Define, in a neighborhood of each point $y \in \Gamma \subset S$, a coordinate system in such a way that the mapping φ induces in the tangent space a linear transformation for which the vector $l(x)$ would be an eigenvector. Together with the natural coordinate systems inside D and D', the introduced coordinate systems form on the manifold S a differentiable structure of class $C^{(3)}$.

Denote by $\sigma(x) = \{\sigma_i^j(x)\}$ such a matrix that $\{a_{ij}(x)\} = \sigma(x)\sigma^*(x)$. Extend the functions $\sigma_i^j(x)$, $b_i(x)$ to the whole manifold S in such a way that they are invariant with respect to the transformation φ. Next consider on the manifold S the stochastic equation

$$\tilde{x}_t - \tilde{x}_s = \varepsilon \int_s^t \sigma(\tilde{x}_u)\, d\xi_u + \int_s^t b(\tilde{x}_u)\, du. \tag{3}$$

Here $\xi_u^1 = \{\xi_u^1, \dots, \xi_u^n\}$ is an n-dimensional Wiener process, $b(x) = \{b_1(x), \dots, b_n(x)\}$, and the first integral on the right-hand side of equation (3) is understood as a stochastic one ([5]). As follows from ([3]), equation (3) has a solution. From this solution we construct a Markov process ([1]) $\widetilde{X}^\varepsilon = \{\tilde{x}_t^\varepsilon, \widetilde{P}_x^\varepsilon\}$. Let $\tau^\varepsilon = \inf\{t : \tilde{x}_t^\varepsilon \in \Gamma_1\}$. Define the random function $x_t^\varepsilon(w)$ as follows:

$$x_t^\varepsilon(\omega) = \begin{cases} \tilde{x}_t^\varepsilon(\omega), & \text{if } t < \tau^\varepsilon \text{ and } \tilde{x}_t^\varepsilon \in D, \\ \varphi(\tilde{x}_t^\varepsilon(\omega)), & \text{if } t < \tau^\varepsilon \text{ and } \tilde{x}_t^\varepsilon \in D', \\ \tilde{x}_{\tau^\varepsilon}^\varepsilon(\omega), & \text{if } t \geq \tau^\varepsilon. \end{cases}$$

Then the measures P_x^ε can be so defined that the pair $X^\varepsilon = \{x_t^\varepsilon, P_x^\varepsilon\}$ forms a Markov process.

Theorem 1. For any $\varepsilon > 0$, the function $u^\varepsilon(x) = M_x \psi(x_{\tau^\varepsilon}^\varepsilon) =$

$$= \int_\Omega \psi(x_{\tau^\varepsilon}^\varepsilon) P_x(d\omega)$$

is a solution of problem (2).

For the construction of the process X^ε and the proof of Theorem 1, see ([4]).

In what follows we shall assume, for simplicity, that $D \subset R^2$. Denote by $H(x)$ the characteristic of the equation $b_1(x)\partial v/\partial x^1 + b_2(x)\partial v/\partial x^2 = 0$, passing through the point x. We introduce the following assumptions:

1. The equation $b_1^2(x) + b_2^2(x) = 0$ determines only a finite number of smooth curves $\delta_1, \dots, \delta_{k-1}$, which divide the domain D into k open sets $\Delta_1, \dots, \Delta_k$. Each curve δ_i joins points $a_i, b_i \in \Gamma$.

2. If $x \in \Delta_i,\ i = 1, 2, \dots, k$, then $H(x)$ reaches the boundary, i.e., there exists a point $H_1(x) \in \Gamma$ such that either for some $t = \hat{t}$, $x_t = H_1(x)$, or

$$\lim_{t\to\infty} x_t = H_1(x).$$

Here x_t is the solution of the system of equations $\dot{x}^i = b_i(x)$ with initial data $x(0) = x$.

3. Denote

$$A = \{y:\ y = H_1(x) \in \Gamma_2\}, \qquad C = \Gamma_1 \cup \{y:\ y \in \Gamma,\ H_1(y) \in \Gamma_1\}.$$

We assume that

$$\overline{A} \setminus A \subset \overline{C} \setminus C.$$

(Each connected component of the set A is "surrounded" by the set C.)

4. For each curve δ_i there is a Δ_j such that

$$\delta_i \in \overline{\Delta}_j \setminus \Delta_j$$

and $H_1(x) \in \Gamma_1$ for $x \in \Delta_j$.

5. If $x \in \Gamma$ and $x \to a_i$, then $H_1(x) = x$ or $H_1(x) \to b_i$.

By virtue of properties 3 and 5, for any curve δ_i one of the points a_i, b_i belongs to the set Γ_1. Put $\bar{a} = a_i$, if $a_i \in \Gamma_1$, and $\bar{a} = b_i$, if $a_i \notin \Gamma_1$. The point $\bar{a}$ always belongs to Γ_1.

Theorem 2. Let $u^\varepsilon(x)$ be a solution of problem (2), and let the point $x \in D$ be such that $H_1(x) \in \Gamma_1$. Then

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = \psi(H_1(x)).$$

The proof of Theorem 2 follows from the fact that the solutions of equation (3) converge, uniformly on any finite interval of time $[0, T]$, as $\varepsilon \to 0$, to the solution of the equation obtained from (3) by putting $\varepsilon = 0$.

In what follows we shall study the case in which $H_1(x) \in \Gamma_2$. Denote by B the connected component of the set A containing the point $H_1(x)$. Write the stochastic equations (3) in the coordinate system * in which the set B is a rectilinear interval (α, β) on the x-axis, $\alpha < \beta$, and the vector field $l(x)$ is normal to B:

$$x_t - x_0 = \varepsilon \int_0^t \sigma_{11}(x_u, y_u)\, d\xi_u^1 + \varepsilon \int_0^t \sigma_{12}(x_u, y_u)\, d\xi_u^2 + \int_0^t b_1(x_u, y_u)\, du, \quad (4)$$

* We assume for simplicity that in this coordinate system the stochastic

equations do not contain terms of the form

$$\varepsilon^2 \int_0^t f(x_u)\,du.$$

$$y_t - y_0 = \varepsilon \int_0^t \sigma_{21}(x_u, y_u)\,d\xi_u^1 + \varepsilon \int_0^t \sigma_{22}(x_u, y_u)\,d\xi_u^2 + \int_0^t b_2(x_u, y_u)\,du. \quad (4)$$

Theorem 3. Let $b_1(x,0) \neq 0$ for $x \in (\alpha, \beta)$. Then, if $b_1(x,0) > 0$, then

$$\lim_{\varepsilon \to \infty} u^\varepsilon(x) = \psi(\bar{\beta});$$

if $b_1(x,0) < 0$, then

$$\lim_{\varepsilon \to \infty} u^\varepsilon(x) = \psi(\bar{\alpha}).$$

Theorem 4. Suppose that at the points of the set B the characteristics are tangent to the vectors $l(x)$. Suppose, moreover, that $b_2(x,0) \neq 0$ for $x \in B$;

$$\lim_{x \to \alpha} \frac{1}{b_2(x,0)} \frac{\partial b_1(x,0)}{\partial y} < M, \qquad \lim_{x \to \beta} \frac{1}{b_2(x,0)} \frac{\partial b_1(x,0)}{\partial y} \geq -M.$$

Then

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = u(x),$$

where $u(x)$ is the solution of the boundary-value problem

$$[\sigma_{11}^2(x,0) + \sigma_{12}^2(x,0)] \frac{d^2 u}{dx^2} - \frac{\sigma_{21}^2(x,0) + \sigma_{22}^2(x,0)}{b_2(x,0)} \frac{\partial b_1(x,0)}{\partial y} \frac{du}{dx} = 0, \quad (5)$$

$$u(\alpha) = \psi(\alpha), \qquad u(\beta) = \psi(\beta).$$

The proof of Theorem 4 is carried out with the aid of the following lemmas.

Lemma 1. Denote

$$\tilde{\tau}^\varepsilon = \inf\{t : x_t^\varepsilon \notin (\hat{\alpha}, \hat{\beta})\}; \qquad \bar{\tau}^\varepsilon = \inf\{t : z_t^\varepsilon \notin (\hat{\alpha}, \hat{\beta})\},$$

where $(\hat{\alpha}, \hat{\beta}) \subset (\alpha, \beta)$, and z_t^ε is the solution of the following equation*

$$z_t^\varepsilon - z_0 = \varepsilon \int_0^t \sqrt{\sigma_{11}^2(z_u^\varepsilon, 0) + \sigma_{12}^2(z_u^\varepsilon, 0)}\, d\tilde{\xi}_u - \frac{\varepsilon^2}{2} \int_0^t \frac{\sigma_{21}^2(z_u^\varepsilon, 0) + \sigma_{22}^2(z_u^\varepsilon, 0)}{b_2(z_u^\varepsilon, 0)} \frac{\partial b_1^*}{\partial y}(z_u^\varepsilon, 0)\,du. \quad (6)$$

Then, for any $T > 0$, $\delta > 0$,

$$\lim_{\varepsilon \to 0} P\left\{ \sup_{u < \min(T/\varepsilon^2, \tilde{\tau}^\varepsilon, \bar{\tau}^\varepsilon)} |x_u^\varepsilon - z_u^\varepsilon| > \delta \right\} = 0.$$

We outline the proof of Lemma 1. Denote by $\bar{x}_u^\varepsilon$ and $\bar{z}_u^\varepsilon$ the solutions of the stochastic equations (4) and (6), where the function in $b_2(x,y)$ has been replaced by some function $\bar{b}_2(x,y)$ coinciding with $b_2(x,y)$ in some neighborhood of the interval (α,β) and everywhere in D different from zero. It is proved that, for any $\delta > 0$,

$$\lim_{\varepsilon\to 0} P\left\{\sup_{s<\min(\bar{\tau}^\varepsilon, T/\varepsilon^2)} |y_s| > \delta\right\} = 0. \tag{7}$$

Put

$$c(x) = \frac{1}{\bar{b}_2(x,0)} \frac{\partial b_1(x,0)}{\partial y}.$$

Using Itô' s formula(2) for the function $f(x,y) = y^2 c(x)$ and equality (7), one can prove that

$$\int_0^{s(\varepsilon)} b_1(x_u, y_u)\,du = -\frac{\varepsilon^2}{2}\int_0^{s(\varepsilon)} [\sigma_{21}^2(x_u,0) + \sigma_{22}^2(x_u,0)]c(x_u)\,du + A(s(\varepsilon),\varepsilon), \tag{8}$$

where the random variable $A(s(\varepsilon),\varepsilon)$ tends to zero in the mean square as $\varepsilon \to 0$ and $s(\varepsilon) < T/\varepsilon^2$. Subtracting (6) from (4), and using (8), we obtain that the function

$$m(s) = M|\bar{x}_s - \bar{z}_s|^2$$

satisfies the inequality

$$m(t) \le k\varepsilon^2 \int_0^t m(s)\,ds + MA^2(t,\varepsilon). \tag{9}$$

The constant k depends only on the upper bound of the coefficients of the equation and their derivatives. From (9) it follows that

$$m(t) \le MA^2(t,\varepsilon)e^{k\varepsilon^2 t}.$$

From po-

* Here $\tilde{\xi}_u$ is some Wiener process which depends on $\xi_u^1, \xi_u^2, \varepsilon$.

from the last inequality it follows that $\sup_{t\le T/\varepsilon^2} m(t) \to 0$ as $\varepsilon \to 0$. Using now Kolmogorov' s inequality for martingales, we obtain that

$$\lim_{\varepsilon\to 0} P\left\{\sup_{u<T/\varepsilon^2} |\bar{x}_u - \bar{z}_u| > \delta\right\} = 0.$$

The assertion of Lemma 1 follows from the last relation, if one observes that for $u \le \min(\tilde{\tau}^\varepsilon, \tau^\varepsilon)$ the equalities $\bar{x}_u^\varepsilon = x_u^\varepsilon$, $\bar{z}_u^\varepsilon = z_u^\varepsilon$ are valid.

Lemma 2. *For any $\delta_1, \delta_2 > 0$ there exists δ_3 such that, if $|x - a| < \delta_3$, then*

$$P\left\{|x_{\tau^\varepsilon}^\varepsilon - \alpha| > \delta_1\right\} < \delta_2$$

for all $\varepsilon > 0$.

Theorem 5. *Suppose that at all points of the interval* (α, β)

$$\partial b_1(x,0)/\partial y \neq 0, \qquad b_2(x,0) = 0.$$

Then

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = \psi(\bar{\alpha}),$$

if $\partial b_1(x,0)/\partial y < 0$*;*

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = \psi(\bar{\beta}),$$

if $\partial b_1(x,0)/\partial y > 0$.

We outline the plan of the proof of Theorem 5. With the aid of equality (7) and Itô' s formula (^{2}), it is established that there exists a function $t = t(\varepsilon)$ such that

$$\lim \varepsilon^2 t(\varepsilon) = 0$$

and

$$P\left\{\int_0^{t(\varepsilon)} b_1(x_u, y_u)\,du > N\right\} \to 0$$

for any $N > 0$. Applying further Kolmogorov' s inequality to the martingale

$$x_{t(\varepsilon)} - \int_0^{t(\varepsilon)} b_1(x_u, y_u)\,du,$$

we are convinced that, with probability tending to 1 as $\varepsilon \to 0$, the trajectory visits during the time interval $[0, t(\varepsilon)]$ any neighborhood of the point β, if $\partial b_1(x,0)/\partial y > 0$, or of the point α, if $\partial b_1(x,0)/\partial y < 0$. The proof of the theorem is completed by applying Lemma 2.

The following theorem summarizes and generalizes Theorems 4 and 5.

Theorem 6. *Suppose that the following conditions are satisfied:*

1. *There exist partial derivatives*

$$b_{1y}^{(i)}(x,y) = \partial^i b_1(x,y)/\partial y^i, \quad i \le k+1;$$

$$b_{1y}^{(i)}(x,y) = 0, \quad \text{for } (x,y) \in B \text{ and } i \le k-1,$$

$$b_1^k(x,y) \neq 0, \quad \text{for } (x,y) \in B.$$

2. *There exist partial derivatives*

$$b_{2y}^{(i)}(x,y) = \partial^i b_2(x,y)/\partial y^i, \quad i \le l+1;$$

$$b_{2y}^{(i)}(x,y) = 0 \quad \text{for } (x,y) \in B,\ i < l-1;$$

$$b_{2y}^{(l)}(x,y) = 0 \quad \text{for } (x,y) \in B.$$

Then, if $k > l$ and

$$\left|b_{1y}^{(k)}(x,0)/b_{2y}^{(l)}(x,0)\right| < M,$$

then

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = u(x),$$

where $u(x)$ is the solution of the following problem:

$$[\sigma_{11}^2(x,0)+\sigma_{12}^2(x,0)]\,\frac{d^2u}{dx^2} - \frac{b_{1y}^{(k)}(x,0)}{b_{2y}^{(l)}(x,0)}\,[\sigma_{21}^2(x,0)+\sigma_{22}^2(x,0)]\,\frac{du}{dx} = 0,$$

$$u(\alpha) = \psi(\bar\alpha), \qquad u(\beta) = \psi(\bar\beta).$$

If $k \le l$, then $u^\varepsilon(x)$ tends as $\varepsilon \to 0$ to $\psi(\bar\alpha)$ or to $\psi(\bar\beta)$, if $b_{1,y}^{(k)}(x,0)$ is respectively positive or negative.

Remark 1. In exactly the same way one treats the case when the operator L_1 in equality (1) contains first derivatives.

The author expresses gratitude to E. B. Dynkin for his attention to the present work.

Moscow State University
named after M. V. Lomonosov

Received
8 XII 1961

References

1. E. B. Dynkin, *Foundations of the Theory of Markov Processes*, 1961.
2. K. Itô, *Nagoya Math. J.*, **3**, 55 (1951).
3. I. V. Girsanov, *DAN*, **138**, No. 1 (1961).
4. M. I. Freidlin, *Proceedings of the Fourth All-Union Conference on Probability Theory and Mathematical Statistics*, Vilnius, 1960.
5. J. Doob, *Stochastic Processes*, IL, 1956.

Note: Figure translations are in progress. See original paper for figures.