A. A. KIRILLOV

1962

SovietRxiv

Abstract

Full Text

A. A. KIRILLOV

ON INFINITE-DIMENSIONAL REPRESENTATIONS OF THE FULL MATRIX GROUP

(Presented by Academician I. G. Petrovskii, 28 XII 1961)

Let G be the group of all nonsingular n-th order matrices (real or complex). Denote by G_0 the subgroup of G consisting of matrices whose last row has the form $(0, 0, \dots, 0, 1)$. In the work of I. M. Gelfand and M. A. Naimark ([1]) it was proved, for the complex group G, that

Proposition 1. *Every unitary irreducible representation of the group G remains irreducible if it is regarded as a representation of the subgroup G_0.*[1]

The proof given in ([1]) is based on the fact that explicit formulas are known for all unitary irreducible representations of the complex group G.

In the present note we give a new proof of Proposition 1, which does not use the explicit form of the representations and is equally applicable to both the complex and the real group G. Since the representations of the group G_0 are comparatively simple in structure, this result may in turn serve as a basis for finding the explicit form of the representations of the group G.

Our proof is based on the following remarkable property of the subgroup G_0. We shall call two elements g_1 and g_2 of G **conjugate relative to** G_0, if for some $g_0 \in G_0$ the equality $g_1 = g_0 g_2 g_0^{-1}$ holds. The whole group G is thereby decomposed into classes of elements conjugate relative to G_0. It turns out that this decomposition almost coincides with the ordinary decomposition of G into conjugacy classes. For a more precise formulation, introduce the following notation. If G is the complex group, denote by Λ the set of all matrices in G of the form

$$\left\| \begin{matrix} \lambda_1 & & & & \\ & \ddots & & 0 & \\ & & \ddots & & \\ 0 & & & \lambda_{n-1} & \\ 1 & 1 & \cdots & 1 & \lambda_n \end{matrix} \right\|, \quad \text{where } |\lambda_1| < |\lambda_2| < \dots < |\lambda_n|.$$

If G is the real group, set

$$\Lambda = \Lambda_1 + \dots + \Lambda_{[\frac{n}{2}]},$$

[1]More precisely, in ([1]) the group G is taken to be the group of complex matrices with determinant 1, and G_0 the subgroup of matrices for which $a_{nk} = 0$, $k = 1, 2, \dots, n-1$. However, Proposition 1 is easily derived from the results of ([1]).

where Λ_k is the set of all matrices in G of the form

$$\left\| \begin{array}{cccccc} \Phi_1 & & & & 0 & \\ & \ddots & & & & \\ & & \Phi_k & & & \\ & & & \lambda_1 & & \\ 0 & & & & \ddots & \\ & & & & & \lambda_{n-2k} \\ 1 & 1 & \cdots & \cdots & 1 & \end{array} \right\| \quad \text{for } k \neq \frac{n}{2}.$$

and of the form

$$\left\| \begin{array}{cccc} \Phi_1 & & & 0 \\ & \ddots & & \\ 0 & & \ddots & \\ E & & & E\,\Phi_k \end{array} \right\| \quad \text{for } k = \frac{n}{2}.$$

Here

$$\Phi_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the numbers $\lambda_i, \alpha_i, \beta_i$ satisfy the inequalities

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{n+2k}, \qquad \alpha_1^2 + \beta_1^2 < \alpha_2^2 + \beta_2^2 < \cdots < \alpha_k^2 + \beta_k^2.$$

Obviously, the set Λ contains no more than one element from each class of conjugate elements in G. At the same time, it can be shown that almost every matrix in G can be represented in the form $g = g_0 \lambda g_0^{-1}$, $g_0 \in G_0$, $\lambda \in \Lambda$, i.e. can be brought to canonical form by a transformation from G_0. For what follows the following formulation of this fact is convenient.

Theorem 1. *The mapping $\gamma : G_0 \times \Lambda \to G$, defined by the formula $\gamma(g_0, \lambda) = g_0 \lambda g_0^{-1}$, is a homeomorphism of $G_0 \times \Lambda$ onto a certain domain $\Omega \subset G$. The complement of Ω in G is the union of a finite number of manifolds of lower dimension and, consequently, has measure zero.*

Thus, with the aid of the subgroup G_0 one obtains a convenient parametrization of the group G as a set of classes of conjugate elements.

Theorem 2. *If K is a class of conjugate elements in G, containing the matrix $\lambda \in \Lambda$, then for any finite bounded function φ on G the equality*

$$\int_K \varphi(g)\, d\mu_K(g) = c \int_{G_0} \varphi(g_0 \lambda g_0^{-1})\, d\mu_l(g_0),$$

holds, where $d\mu_K(g)$ is an invariant (with respect to inner automorphisms of G) measure on K; $d\mu_l(g_0)$ is a left-invariant measure on the group G_0; c is a normalizing constant independent of φ.

For the proof of Proposition 1 it suffices to prove

Proposition 2. *Let T be an irreducible representation of G in the space H. Every bounded operator P in the space H which commutes with the operators T_{g_0}, $g_0 \in G_0$, commutes with all operators of the representation.*

Proof. We first consider the case of a finite-dimensional representation. Let χ be the function on G defined by the formula $\chi(g) = \mathrm{sp}(PT_g)$, where P is an operator in H commuting with T_{g_0}, $g_0 \in G_0$. Then

$$\chi(g_0 g g_0^{-1}) = \mathrm{sp}(PT_{g_0}T_gT_{g_0}^{-1}) = \mathrm{sp}(T_{g_0}PT_gT_{g_0}^{-1}) = \mathrm{sp}(PT_g) = \chi(g),$$

i.e. the function $\chi(g)$ is constant on classes of elements conjugate with respect to G_0. From what was said above and from the continuity of $\chi(g)$ it follows that this function is constant on the ordinary classes of conjugate elements. Hence

$$\mathrm{sp}(PT_{g_1}) = \chi(g_1) = \chi(gg_1g^{-1}) = \mathrm{sp}(PT_gT_{g_1}T_g^{-1}) = \mathrm{sp}[(T_g^{-1}PT_g)T_{g_1}].$$

Therefore

$$\mathrm{sp}[(P - T_g^{-1}PT_g)T_{g_1}] = 0 \tag{1}$$

for all $g, g_1 \in G$. But if the representation T is irreducible, then the operators T_{g_1}, $g_1 \in G$, generate the space of all operators in H. Consequently, equality (1) is possible only when $P - T_g^{-1}PT_g = 0$, i.e. when the operator P commutes with all operators of the representation.

In the case of an infinite-dimensional representation the proof is based on the same idea, but is technically more complicated. We shall indicate the main stages of this proof. First we introduce the following terminology and notation. By a generalized function on a manifold M we shall mean a continuous linear functional on the space $D(M)$ of infinitely differentiable finite functions on M. If φ is a function on G, P is an operator in H, and Ω is a subset of G, put $\varphi^g(g_1) = \varphi(gg_1g^{-1})$,

$P^g = T_gPT_g^{-1}$, $\Omega^g = g^{-1}\Omega g$. Finally, denote by χ the generalized function on G defined by the formula $(\chi, \varphi) = \mathrm{sp}(PT_\varphi)$, where $T_\varphi = \int \varphi(g)T_g\,dg$, $\varphi \in D(G)$, and P is an operator in H that commutes with T_{g_0}, $g_0 \in G_0$. Obviously, for any $g_0 \in G_0$ the equality $(\chi, \varphi) = (\chi, \varphi^{g_0})$ holds.

Lemma 1. *Let Ω and Λ be the same as in Theorem 1. There exists a generalized function χ_1 on Λ such that, for any function $\varphi \in D(\Omega)$, the equality*

$$(\chi, \varphi) = (\chi_1, \varphi_1), \qquad \text{where} \quad \varphi_1(\lambda) = \int_{G_0} \varphi(g_0 \lambda g_0^{-1})\, d\mu_l(g_0).$$

holds.

The proof of this lemma is based on the fact that every generalized function on a Lie group that is invariant with respect to left translations coincides, up to a factor, with a left-invariant measure.

Lemma 2. *If $\varphi \in D(\Omega \cap \Omega^g)$, then $(\chi, \varphi) = (\chi, \varphi^g)$.*

Proof. By Lemma 1, $(\chi, \varphi) = (\chi_1, \varphi_1)$, $(\chi, \varphi^g) = (\chi_1, \varphi_1^g)$. But $\varphi_1 = \varphi_1^g$, since the integrals of the functions φ and φ^g over any class of conjugate elements coincide, and the value of $\varphi_1(\lambda)$ is, by Theorem 2, equal to the integral of φ over the class of conjugate elements containing λ.

Lemma 3. *If Ω is an open subset of G whose complement has measure zero, and if A is a bounded operator in H satisfying the condition $\mathrm{sp}(AT_\varphi) = 0$ for all $\varphi \in D(\Omega)$, then $A = 0$.*

Proof. Replacing, if necessary, A by AT_g, $g \in \Omega$, we may assume that Ω contains some neighborhood of the identity in G. As is known (see, for example, ([2])), the space H decomposes into a direct sum of finite-dimensional spaces H_c, in each of which the restriction of the representation T to the maximal compact subgroup K of the group G is a multiple of the irreducible representation c. If $\xi_1, \dots, \xi_{n(c)}$ is a basis in the space of the representation c, then H_c naturally decomposes into a sum of spaces $H_{c\xi}$, $\xi = \xi_1, \dots, \xi_{n(c)}$. In the work of F. A. Berezin ([3]) the following fact is proved. If B is any operator in $H_{c_0\xi_0}$, and W is an arbitrarily small neighborhood of the identity in G such that $K \cdot W \cdot K = W$, then in $D(W)$ there exists a function φ such that T_φ coincides with B in $H_{c_0\xi_0}$ and is equal to zero on the remaining $H_{c\xi}$.

It follows from this that the space of operators T_φ, $\varphi \in D(\Omega)$, contains all finite-dimensional operators in H that are zero on all $H_{c\xi}$ except $H_{c_1\xi_1}$, with range in $H_{c_2\xi_2}$. Indeed, since the space $D(\Omega)$ is dense in $L^1(G)$, the set of operators T_φ, $\varphi \in D(\Omega)$, is dense in the space of all bounded operators in H with respect to the strong operator topology. Therefore, for any pair $H_{c_1\xi_1}$, $H_{c_2\xi_2}$ there is an operator T_φ, $\varphi \in D(\Omega)$, for which $P_{c_2\xi_2} T_\varphi P_{c_1\xi_1}$ has the maximal possible rank (here $P_{c\xi}$ is the projection operator onto $H_{c\xi}$). Let W be a sufficiently small neighborhood of the identity, having the property $K \cdot W \cdot K = W$, such that $\alpha * \varphi * \beta \in D(\Omega)$ for any $\alpha, \beta \in D(W)$. Choosing appropriate α and β, one can obtain the equality $T_{\alpha*\varphi*\beta} = B$, where B is any previously prescribed operator from $H_{c_1\xi_1}$ to $H_{c_2\xi_2}$. The assertion of the lemma follows without difficulty.

Proposition 2 follows directly from Lemmas 2 and 3. Indeed, for all $\varphi \in D(\Omega \cap \Omega^g)$, by Lemma 2 the equality

$$\mathrm{sp}(PT_\varphi) = \mathrm{sp}(PT_{\varphi^g}) = \mathrm{sp}(PT_{g^{-1}} T_\varphi T_g) = \mathrm{sp}(P^g T_\varphi),$$

holds, i.e. $\mathrm{sp}[(P-P^g)T_\varphi]=0$. Since the complement of $\Omega\cap\Omega^g$ in G has measure zero, Lemma 3 gives $P-P^g=0$; consequently, P commutes with all operators of the representation.

The author considers it his duty to express gratitude to F. A. Berezin for discussing the contents of the present note.

Moscow State University
named after M. V. Lomonosov

Received
23 XII 1961

References

1. I. M. Gelfand, M. A. Naimark, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **36**, §§ 5, 14, 17 (1950).

2. R. Godement, *Sborn. per. Matematika*, **5**, 5, 55 (1961).

3. F. A. Berezin, *Tr. Mosk. Matem. obshch.*, **6**, no. 5, 443 (1957).

Note: Figure translations are in progress. See original paper for figures.