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**Abstract**

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**MATHEMATICS**

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# ON THE QUESTION OF THE ZERO STABILITY ZONE

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1. Consider the equation

$$\ddot{x} + q(t)x = 0. \tag{1}$$

For simplicity the function $q(t)$ is assumed to be continuous; with the usual reservations, everything that follows remains valid for any integrable $q(t)$.

We shall need the following proposition, which is essentially a lower estimate for the distance between zeros of nontrivial solutions of equation (1). Here and below the notation $p_+(t) = \max\{0, p(t)\}$ is used.

**Theorem 1.** *If for some $a$, $0 \leq a < \pi/\omega$, and some natural number $n$ the inequality*

$$\int_t^{t+\omega/n} (q - a^2)_+ \, dt \leq 2a \, \frac{\cos(\omega a/n) - \cos(\pi/n)}{\sin(\omega a/n)} \tag{2}$$

*holds for all $t$ in the interval $0 \leq t \leq \omega(1 - 1/n)$, then every nontrivial solution of equation (1) has no more than one zero in the interval $[0, \omega]$.*

In the proof of this theorem, in particular, the integral comparison principle (6) is used. Also very useful here is the consideration of equations of the form $\ddot{x}+p(t)x = 0$ with a "generalized"coefficient $p(t)$, which is the sum of a continuous (or integrable) function and a certain number of delta-functions.

2. We now turn to the question of the stability of the solutions of equation (1). In what follows it is assumed that $q(t)$ is $\omega$-periodic and satisfies the condition

$$\int_0^{\omega} (qr^2 - \dot{r}^2)\, dt \geq 0, \tag{3}$$

where $r(t)$ is some absolutely continuous function such that $|r(0)| = |r(\omega)|$; if $r(t)$ does not change sign on $[0, \omega]$ and $\dot{r}(0) = \dot{r}(\omega)$, then it is additionally required that $r(t)$ not be a solution of equation (1). Condition (3) is satisfied, in particular (for $r(t) \equiv 1$), if $q(t)$ $(\not\equiv 0)$ is nonnegative on average.

With the aid of Theorem 1 the following proposition is easily established.

**Theorem 2.** *If for some $a$, $0 \leq a \leq \pi/\omega$, and some natural number $n$ inequality (2) holds for every $t$, $0 \leq t < \omega$, then all solutions of equation (1), together with their derivatives, are bounded on $(-\infty, \infty)$.*

The stability criterion for the trivial solution formulated here corresponds to the zero stability zone in the sense of Lyapunov. Theorem 1 can obviously also be used to obtain criteria corresponding to other stability zones. We shall not dwell on this in greater detail, since the main difficulty lies precisely in Theorem 1.

3. In the limiting case $a = 0$, condition (2) takes the form

$$\int_t^{t+\omega/n} q_+(t)\, dt \leq \frac{c_n}{\omega}, \qquad c_n = 2n\left(1 - \cos\frac{\pi}{n}\right). \tag{4}$$

We find

$$c_1 = 4, \qquad c_2 = 4, \qquad c_3 = 3, \qquad c_4 = 8 - 4\sqrt{2}, \ldots$$

As $n \to \infty$, the constants $c_n$ decrease asymptotically as $\pi^2/n$, so that in the limit condition (4) (as well as the general condition (2)) passes into the well-known criterion of N. E. Zhukovskii ([2]) for the zero stability zone: $q(t) \leq \pi^2/\omega^2$.

The coincidence of $c_1$ and $c_2$ entails an interesting consequence: it turns out that, instead of the classical condition of A. M. Lyapunov ([1])–M. G. Krein ([4])

$$\int_0^{\omega} q_+(t)\, dt \leq \frac{4}{\omega},$$

it is sufficient to require fulfillment of the less stringent condition

$$\int_t^{t+\omega/2} q_+(t)\, dt \leq \frac{4}{\omega} \qquad (0 \leq t < \omega). \tag{5}$$

An analogous situation also occurs in the general case: for $n = 1$, Theorem 2 passes into the stability criterion contained in the works of V. A. Yakubovich ([3]) and M. G. Krein ([4]):

$$\int_0^{\omega} (q - a^2)_+ \, dt \le 2a \operatorname{ctg} \frac{\omega a}{2};$$

putting, however, $n = 2$, we see that in this criterion the integral over the period can also be replaced by an integral over each half-period.

Thus, the case $n = 1$, when Theorem 2 passes into known results, turns out to be subordinate to the case $n = 2$; the cases $n = 2, 3, 4, \ldots$ are independent.

For $n = 4$, Theorem 2 strengthens Opial' s result ([5]), who obtained the following stability criterion ($0 \le a \le \pi/\omega$):

$$\int_t^{t+\omega/4} (q - a^2)_+ \, dt \le \frac{\omega a}{2\pi} \operatorname{ctg} \frac{\omega a}{4} \left( 3 - \sqrt{4 \operatorname{tg}^2 \frac{\omega a}{4} + 5} \right) \qquad (0 \le t < \omega). \quad (6)$$

It is easy to see that, for any admissible value of $a$, the right-hand side of inequality (2) for $n = 4$ exceeds the right-hand side of inequality (6) by no less than a factor of $4/3$. In particular, for $c_4$ in ([5]) the value $6 - 2\sqrt{5}$ is indicated instead of the exact value $8 - 4\sqrt{2}$. Opial' s assertion on the non-improvability of the results he obtained is therefore erroneous.

4. For those cases in which verification of condition (2), or of more particular conditions of the same type, is difficult, it is useful to note that in reality only certain values of $t$ from $[0, \omega)$ require verification. If, for example, it is necessary to verify fulfillment of condition (5) for some nonnegative differentiable $q(t)$, then it is sufficient to restrict oneself to those $t$ for which $q(t + \omega/2) = q(t)$, $\dot{q}(t + \omega/2) \le \dot{q}(t)$. Indeed, we are interested only in the points of maxima of the periodic function

$$Q(t) = \int_t^{t+\omega/2} q \, dt,$$

but

$$\dot{Q}(t) = q(t + \omega/2) - q(t).$$

Thus one may expect that, in the majority of practically occurring cases, verification will be required only for one or several values of $t$.

The conditions of Theorems 1 and 2 involve integrals over intervals of length $\omega/n$. Analogous propositions can be given for intervals of arbitrary length $l$

(only the case $l \leq \omega/2$ is of interest), but the formulations then become more cumbersome.

5. In conclusion we present a result of a somewhat different character. Define the constant $k_0$ ($k_0 \simeq 7.85$) by the relations

$$k_0 = \gamma^2 - \gamma \operatorname{tg} \gamma, \qquad 2\gamma = \operatorname{tg} 2\gamma \quad \left(\frac{\pi}{2} < \gamma < \pi\right).$$

**Theorem 3.** *If $p_+(t)$ is continuous and monotone on $[0, \omega]$ and*

$$\int_0^\omega p_+(t)\, dt \leq \frac{k_0}{\omega}, \tag{7}$$

*then every nontrivial solution of the equation $\ddot{x} + p(t)x = 0$ has no more than one zero in the interval $[0, \omega]$.*

This proposition is proved with the aid of the integral comparison principle ((6), Theorem 2). As is known [(2)], an analogous assertion is valid without the assumption of monotonicity of $p_+(t)$, if in inequality (7) $k_0$ is replaced by the constant 4.

Theorem 3 can evidently be reformulated as a lower estimate for the first positive eigenvalue of the boundary-value problem

$$\ddot{x} + \lambda p(t)x = 0, \qquad x(0) = x(\omega) = 0 \tag{8}$$

in the case of monotone $p_+(t)$. Along these lines one can also establish estimates for other eigenvalues of problem (8), which prove useful for obtaining stability criteria for solutions of equation (1) in the case of periodic piecewise-monotone $q(t)$.

6. All the results presented are sharp, i.e., they cannot be improved without additional assumptions.

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*Note: Figure translations are in progress. See original paper for figures.*