Mathematics

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1962

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Abstract

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INVESTIGATION OF A SINGULARLY PERTURBED AUTONOMOUS SYSTEM IN A NEIGHBORHOOD OF A TORUS

(Presented by Academician N. N. Bogolyubov, 12 III 1962)

We consider the system of differential equations

$$\frac{dx}{dt} = f(x,u), \qquad \frac{dy}{dt} = f^*(y,v), \qquad \varepsilon\frac{dv}{dt} = F(x,u), \qquad \varepsilon\frac{du}{dt} = F^*(y,v), \quad (1)$$

which is singularly perturbed with respect to the system

$$\frac{d\bar{x}}{dt} = f(\bar{x},\varphi(\bar{x})), \qquad \frac{d\bar{y}}{dt} = f^*(\bar{y},\psi(\bar{y})), \qquad \bar{u} = \varphi(\bar{x}), \qquad \bar{v} = \psi(\bar{y}), \quad (2)$$

where x and f, y and f^*, v and F, u and F^* are vectors of dimensions k, l, m, and n, respectively, and $u = \varphi(x)$ and $v = \psi(y)$ are isolated solutions of the systems $F(x,u) = 0$ and $F^*(y,v) = 0$.

It is assumed that the unperturbed system (2) admits a solution of the form

$$\bar{x} = \bar{x}^0(\theta), \qquad \bar{y} = \bar{y}^0(\vartheta), \qquad \bar{u} = \varphi(\bar{x}^0) = \bar{u}^0(\theta), \qquad \bar{v} = \psi(\bar{y}^0) = \bar{v}^0(\vartheta), \quad (3)$$

periodic in $\theta = \omega t$ and $\vartheta = \nu t$ with period 2π.

With respect to the right-hand sides of system (1), and also the vectors φ and ψ, we shall assume the following.

In the region $x \in U_{\rho_1}$, $y \in U_{\rho_2}$, $u \in U_{\rho_3}$, $v \in U_{\rho_4}$, where $U_{\rho_1}, U_{\rho_2}, U_{\rho_3}, U_{\rho_4}$ denote the ρ_1-, ρ_2-, ρ_3- and ρ_4-neighborhoods of the periodic solution $\bar{x}^0, \bar{y}^0, \bar{u}^0, \bar{v}^0$, the vectors f, f^*, F, F^*, φ, and ψ, together with their partial derivatives with respect to all arguments up to order $(p+1)$ inclusive, are bounded and uniformly continuous.

It is established that system (1) admits a unique integral manifold of the form

$$x = x^0(\theta, \vartheta, \varepsilon), \qquad y = y^0(\theta, \vartheta, \varepsilon), \qquad u = u^0(\theta, \vartheta, \varepsilon), \qquad v = v^0(\theta, \vartheta, \varepsilon), \quad (4)$$

and the properties of this manifold are considered.

By slightly modifying the author' s theorem ([4]), we can assert that if the real parts of the eigenvalues of the matrices $F_u(x, \varphi(x))$ and $F_v^*(y, \psi(y))$ are negative, then system (1) has a unique stable integral manifold of the form

$$u = \varphi(x) + \varphi^*(x, y, \varepsilon), \qquad v = \psi(y) + \psi^*(x, y, \varepsilon), \tag{5}$$

where $\varphi^*(x, y, \varepsilon) \to 0$, $\psi^*(x, y, \varepsilon) \to 0$ as $\varepsilon \to 0$. On this manifold system (1) is rewritten in the form

$$\frac{dx}{dt} = f(x, \varphi(x) + \varphi^*(x, y, \varepsilon)), \qquad \frac{dy}{dt} = f^*(y, \psi(y) + \psi^*(x, y, \varepsilon)), \tag{6}$$

whose order is $m + n$ units less than the order of system (1). For the reduced system (2), the equations in variations corresponding to solution (3) have the form

$$\frac{d\delta\bar{x}}{dt} = \bar{f}(\bar{x}^0)\,\delta\bar{x}, \qquad \frac{d\delta\bar{y}}{dt} = \bar{f}^*(\bar{y}^0)\,\delta\bar{y}, \tag{7}$$

where the matrices $\bar{f} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial u}\dfrac{d\varphi}{dx}$ and $\bar{f}^* = \dfrac{\partial f^*}{\partial y} + \dfrac{\partial f^*}{\partial v}\dfrac{d\psi}{dy}$ are periodic in θ and ϑ with period 2π.

We shall assume that $k-1$ characteristic exponents of the first of equations (7) and $l-1$ characteristic exponents of the second of equations (7) have negative real parts. The first and the second of equations (7) each have one characteristic exponent equal to zero, since $\partial\bar{x}^0/\partial\theta$ and $\partial\bar{y}^0/\partial\vartheta$ are solutions of equations (7).

According to A. M. Lyapunov' s theorem ([1]) and the remarks made in ([3]), there exist real nonsingular matrices $P(\theta)$ and $P^*(\vartheta)$, periodic in θ and ϑ with period $*\,2\pi$, having continuous first derivatives with respect to θ and ϑ, and also square real matrices H and H^*, such that the transformation

$$\delta\bar{x} = \frac{\partial\bar{x}^0}{\partial\theta}\rho + P(\theta)q, \qquad \delta\bar{y} = \frac{\partial\bar{y}^0}{\partial\vartheta}r + P^*(\vartheta)s \tag{8}$$

reduces equations (7) to the equivalent systems with constant coefficients

$$\frac{d\rho}{dt} = 0, \qquad \frac{dq}{dt} = Hq;$$

$$\frac{dr}{dt} = 0, \qquad \frac{ds}{dt} = H^* s, \tag{9}$$

where the eigenvalues of the matrices H and H^* are equal to the nonzero characteristic exponents of equations (7).

We now transform system (6) by means of the substitution

$$x = \bar{x}^0(\theta) + P(\theta)h, \qquad y = \bar{y}^0(\vartheta) + P^*(\vartheta)h^* \tag{10}$$

to the equivalent system

$$\frac{d\theta}{dt} = \omega + A_1(\theta, \vartheta, h, h^*, \varepsilon),$$

$$\frac{d\vartheta}{dt} = \nu + A_2(\theta, \vartheta, h, h^*, \varepsilon),$$

$$\frac{dh}{dt} = Hh + A_3(\theta, \vartheta, h, h^*, \varepsilon),$$

$$\frac{dh^*}{dt} = H^* h^* + A_4(\theta, \vartheta, h, h^*, \varepsilon) \tag{11}$$

of $k + l$ equations with $k + l$ unknowns $\theta, \vartheta, h = (h_2, \dots, h_k), h^* = (h_2^*, \dots, h_l^*)$.

The functions A_i $(i = 1, 2, \dots, 4)$ are defined in the domain $\theta \in \Theta$, $\vartheta \in \Omega$, $h \in U_{\rho_1}$, $h^* \in U_{\rho_2}$, $0 \le \varepsilon < \varepsilon^*$, where U_{ρ_1} and U_{ρ_2} denote the ρ_1- and ρ_2-neighborhoods of the points $h = 0$ and $h^* = 0$, possess bounded and uniformly continuous derivatives with respect to θ and ϑ, and are periodic in θ and ϑ with period 2π.

Applying the method of proof set forth in the monograph ([2]), we verify the validity of the following assertions:

1. There exists a positive number $\varepsilon_1 < \varepsilon^*$ such that, for any positive number $\varepsilon < \varepsilon_1$, the system of equations (11) has a unique integral manifold, periodic in θ and ϑ with period 2π, representable by relations of the form

$$h = g(\theta, \vartheta, \varepsilon), \qquad h^* = g^*(\theta, \vartheta, \varepsilon), \tag{12}$$

* The matrices $P(\theta)$ and $P^*(\vartheta)$ will have period 4π in θ and ϑ if there are negative ones among the real characteristic numbers of equations (7).

where the functions g and g^* are defined in the domain $\theta \in \Theta,\ \vartheta \in \Omega$ and satisfy the inequalities

$$\begin{gathered}|g(\theta,\vartheta,\varepsilon)| \leq D(\varepsilon),\\ |g(\theta',\vartheta',\varepsilon)-g(\theta'',\vartheta'',\varepsilon)| \leq \Delta(\varepsilon)(|\theta'-\theta''|+|\vartheta'-\vartheta''|),\\ |g^*(\theta,\vartheta,\varepsilon)| \leq D^*(\varepsilon),\\ |g^*(\theta',\vartheta',\varepsilon)-g^*(\theta'',\vartheta'',\varepsilon)| \leq \Delta^*(\varepsilon)(|\theta'-\theta''|+|\vartheta'-\vartheta''|),\end{gathered} \tag{13}$$

where $D(\varepsilon) \to 0,\ D^*(\varepsilon) \to 0,\ \Delta(\varepsilon) \to 0,\ \Delta^*(\varepsilon) \to 0$ together with ε; g and g^* have bounded and uniformly continuous derivatives with respect to θ and ϑ up to order p, inclusive.

2. There exist positive constants $\varepsilon_0, \gamma_0, C_0, \sigma_0, \gamma_1, C_1, \sigma_1$ ($\sigma_0 < \rho_1,\ \sigma_1 < \rho_2,\ \varepsilon_0 < \varepsilon_1$) such that, for all $\varepsilon < \varepsilon_0$, any real t_0, and any $\theta \in \Theta,\ \vartheta \in \Omega$, there exists a $(k-1)$-dimensional domain U_{σ_0} of points $\{h\}$ and an $(l-1)$-dimensional domain of points $\{h^*\}$ with the properties

$$|h_t - g(\theta_t,\vartheta_t,\varepsilon)| \leq C_0 e^{-\gamma_0(t-t_0)}|h_0 - g(\theta_0,\vartheta_0,\varepsilon)|,$$

$$|h_t^* - g^*(\theta_t,\vartheta_t,\varepsilon)| \leq C_1 e^{-\gamma_1(t-t_0)}|h_0^* - g^*(\theta_0,\vartheta_0,\varepsilon)|,$$

where h_t and h_t^* are arbitrary solutions of system (11); $h_0 = h_t(t_0),\ h_0^* = h_t^*(t_0),\ \theta_0 = \theta_t(t_0),\ \vartheta_0 = \vartheta_t(t_0)$.

Transferring these assertions to system (1), which is equivalent to system (11), we obtain the following theorem:

Theorem. *Under the assumptions indicated above, there exists a number $\varepsilon_0 > 0$ such that, for any positive $\varepsilon < \varepsilon_0$, the following assertions hold:*

1. *System (1) has a unique integral manifold $\mathfrak{M}$, which is a torus in $(k+l+m+n)$-dimensional space.*
2. *This manifold admits the parametric representation*

$$x = \bar{x}^0(\theta) + P(\theta)g(\theta,\vartheta,\varepsilon), \qquad y = \bar{y}^0(\vartheta) + P^*(\vartheta)g^*(\theta,\vartheta,\varepsilon),$$
$$u = \varphi(x) + \varphi^*(x,y,\varepsilon), \qquad v = \psi(y) + \psi^*(x,y,\varepsilon),$$

where the right-hand sides are defined in the domain $\theta \in \Theta,\ \vartheta \in \Omega,\ 0 < \varepsilon < \varepsilon_0$, are periodic in the angular variables θ and ϑ with period 2π, and have bounded and uniformly continuous derivatives with respect to θ and ϑ up to order p, inclusive.

3. *On the manifold $\mathfrak{M}$, system (1) is equivalent to the system*

$$\frac{d\theta}{dt} = \omega + A_1(\theta, \vartheta, g(\theta, \vartheta, \varepsilon), g^*(\theta, \vartheta, \varepsilon), \varepsilon),$$

$$\frac{d\vartheta}{dt} = \nu + A_2(\theta, \vartheta, g(\theta, \vartheta, \varepsilon), g^*(\theta, \vartheta, \varepsilon), \varepsilon),$$

where the right-hand sides are definite functions, periodic in θ and ϑ with period 2π, and possessing bounded and uniformly continuous derivatives with respect to θ and ϑ up to order p, inclusive.

4. *The manifold $\mathfrak{M}$ has the property of attracting solutions of system (1) that are close to it.*

I take this opportunity to express my gratitude to Academician N. N. Bogolyubov for his attention to this work.

Received
1 III 1962

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Note: Figure translations are in progress. See original paper for figures.