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Abstract

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MATHEMATICS

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ON THE SPECTRAL AND METRIC ISOMORPHISM OF CERTAIN NORMAL DYNAMICAL SYSTEMS

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§ 1. Let M be a Lebesgue space with measure m (see (^{1})) and let $\gamma(x)$ be a measurable real-valued function on M. Define in $L^2(M)$ a unitary operator V_γ by the formula

$$(V_\gamma f)(x) = e^{i\gamma(x)} f(x),$$

and denote by $(\gamma(x))$ the quantity $\gamma(x) - 2k\pi$ for

$$(2k-1)\pi \le \gamma(x) < (2k+1)\pi.$$

The following theorem makes it possible to compute the spectral invariants

$$\rho_1 > \rho_2 > \cdots$$

of the operator V_γ (see (^{2}), p. 164) from the metric invariants $G, G_1, G_2, \dots$ of the function (γ), constructed by V. A. Rokhlin (^{3}). These metric invariants are defined for every measurable real-valued function ξ on M and may be described as follows. Let X_n be a set of maximal measure among all measurable sets on which ξ assumes almost all its values no more than n times (such sets exist); G is the distribution function of the function ξ on M, and G_n is the distribution function of the function ξ on X_n (G_n is not necessarily normalized). As proved in (^{3}), $G, G_1, G_2, \dots$ is a complete system of metric invariants of the function ξ.

Theorem 1. The spectral invariants

$$\rho_1 > \rho_2 > \cdots$$

of the operator V_γ are the Hellinger types of the distribution functions

$$G,\ G - G_1,\ G - G_2, \dots,$$

where $G, G_1, G_2, \dots$ are the metric invariants just described for the function (γ).

Corollary 1. In order that the operator V_γ have simple spectrum, it is necessary and sufficient that the function (γ) be one-to-one almost everywhere; in order that the operator V_γ have homogeneous countably multiple spectrum, it is necessary and sufficient that the function (γ) assume almost all its values (with respect to the measure G) an infinite number of times.

Corollary 2. If the measure m is continuous, then the operator V_γ either has no discrete spectrum, or has a countably multiple discrete part.

§ 2. Theorem 1 makes it possible to solve the problem of computing the spectrum of normal dynamical systems (n.d.s.). We use the definitions and notation introduced in (^{4}). Let $F(d\lambda)$ be the continuous spectral measure of a one-dimensional n.d.s. $\{s, \mu, T\}$. Consider the unitary ring

$$\mathcal{L}_F = \sum_{k=0}^{\infty} \oplus L^2_{F(k)}$$

(see (^{4})). The automorphism V of the ring $\mathcal{L}_F$, defined by the formula

$$(Vf_k)(\lambda_1, \dots, \lambda_k) = \exp\left[i \sum_{p=1}^{k} \lambda_p \right] f_k(\lambda_1, \dots, \lambda_k), \qquad f_k \in L^2_{F(k)},$$

is isomorphic to the automorphism U_T of the ring $L^2_\mu(s)$ corresponding to the shift in the space s (see also (^{5})). Therefore, finding the spectrum of an n.d.s. reduces to computing the spectral invariants of the automorphism V of the ring $\mathcal{L}_F$, i.e., to computing the spectral invariants of V on each of the subspaces $L^2_{F(k)}$. If now we take, as M, the set of points

$$x = (\lambda_1, \dots, \lambda_k) \in [-\pi, \pi]^k$$

satisfying the condition

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k;$$

as m, the measure induced on M by the measure

$$F^{(k)} = F \times \dots \times F$$

(k factors); and as $\gamma(x)$, the function on M:

$$\gamma(x) = \sum_{p=1}^{k} \lambda_p,$$

then we obtain the conditions of Theorem 1. The distribution function of the function (γ) is in this case F^k, the k-th convolution of the measure F. Let $\Lambda_k \subset [-\pi, \pi]$ be a set of full F^k-measure ($k \geqslant 2$), and Λ a set of full F-measure. Consider the equation:

$$\lambda_1 + \lambda_2 + \dots + \lambda_k = \lambda; \qquad (*)$$

here $\lambda_i \in \Lambda,\ i = 1, 2, \dots, k;\ \lambda \in \Lambda_k;\ k \geqslant 2$. Denote by $c_k(\lambda)$ the number of distinct solutions of equation $(*)$, i.e., the number of points $x = (\lambda_1, \dots, \lambda_k) \in M$ such that

$$\sum_{p=1}^{k} \lambda_p = \lambda.$$

We shall say that a unitary operator has a spectrum of unbounded multiplicity if the latter is neither of finite multiplicity nor of infinite multiplicity. Application of Theorem 1 leads to the theorem:

Theorem 2. 1) A one-dimensional n.d.s.: a) has simple spectrum if and only if $c_k(\lambda) = 1$ for almost all λ with respect to the measure F^k and for every $k \geqslant 2$; b) has spectrum of unbounded multiplicity if and only if $1 \leqslant c_k(\lambda) < \infty$ for almost all λ with respect to the measure F^k and for every $k \geqslant 2$, and for some k, $c_k(\lambda) > 1$ on a set of positive F^k-measure; c) has infinite-multiplicity spectrum if and only if $c_k(\lambda) = \infty$ for some k on a set of positive F^k-measure.

2) If $c_k(\lambda) = \infty$ for almost all λ with respect to the measure F^k and every $k \geqslant 2$, then the spectral types of the n.d.s. are the following:

$$\sum_{k=0}^{\infty} F^k, \qquad \sum_{k=2}^{\infty} F^k, \qquad \sum_{k=2}^{\infty} F^k$$

(F^0 is the unit mass at zero).

3) Every one-dimensional n.d.s. with a continuous spectral measure can be decomposed into a direct product of one-dimensional n.d.s.' s, each of which satisfies one of the three conditions in item 1).

It follows from the theorem just formulated that a one-dimensional n.d.s. cannot have a finite-multiplicity spectrum different from a simple one. This result is given in (^{6}).

Theorem 2 is easily extended to multidimensional n.d.s.' s. We give some consequences. Let $\|F_{ij}\|$ be the spectral matrix of a multidimensional n.d.s., all elements of which are continuous. Let

$$\Phi = \sum F_{ii}.$$

1. A multidimensional n.d.s. has a homogeneous countable-multiplicity spectrum if and only if

   $$\Phi < \sum_{k=2}^{\infty} \Phi^k.$$

   For one-dimensional n.d.s.' s this result was published by I. V. Girsanov (^{6}). Examples of one-dimensional n.d.s.'s with homogeneous non-Lebesgue spectrum were constructed by B. M. Makarov (unpublished).

2. If the spectrum of an n-dimensional n.d.s. $(n < \infty)$ is inhomogeneous, then it contains an n-fold component.

3. Every n.d.s. with simple spectrum is isomorphic to a one-dimensional n.d.s.
4. An n.d.s. (of any dimension) cannot have a finite-multiplicity spectrum different from a simple one.

§ 3. Let F_0 be Lebesgue measure on $[-\pi, \pi]$. The unitary ring $\mathcal{L}_{F_0} = \mathcal{L}$ is called standard. A normal automorphism of the ring $\mathcal{L}$ is an automorphism defined by the formula

$$(V_\varphi f_k)(\lambda_1, \dots, \lambda_k) = \exp\left[i \sum_{p=1}^{k} \varphi(\lambda_p)\right] f_k(\lambda_1, \dots, \lambda_k), \qquad f_k \in L^2_{F_0(k)};$$

here $\varphi(\lambda)$ is an arbitrary odd measurable function mapping $[-\pi, \pi]$ into itself. In (^{4}) the following theorem is given:

Let $\{S, \mu, T\}$ be a multidimensional n.d.s.; the automorphism U_T of the unitary ring $L^2_\mu(S)$, corresponding to the shift T in S, is isomorphic to some normal automorphism of the ring $\mathcal{L}$.

Application of Theorem 1 makes it possible to establish the converse result.

Theorem 3. *Let V_φ be an arbitrary normal automorphism of the ring $\mathcal{L}$. There exists an n.d.s. $\{S, \mu, T\}$, of at most countable dimension, such that the automorphism U_T of the ring $L^2_\mu(S)$ corresponding to the shift is isomorphic to V_φ. The dimension of this n.d.s. is not less than the multiplicity of the spectrum of V_φ in the subspace $L^2_{F_0^1}$.*

It follows from Corollary 2 that a normal automorphism in whose spectrum there is a discrete component is necessarily nonergodic.

We shall call an automorphism W of the ring $\mathcal{L}$ linear if $WL^2_{F_0^1} = L^2_{F_0^1}$. Clearly, every normal automorphism is linear.

Theorem 4. *Every linear automorphism in whose spectrum the discrete component is either absent or of countable multiplicity is linearly isomorphic to some normal automorphism.*

§ 4. We now turn to the problem of metric isomorphism of n.d.s. From the theorems of A. N. Kolmogorov ((^{7}), p. 23) it follows that one-dimensional n.d.s. with a continuous spectral measure are metrically isomorphic if their spectral measures are mutually absolutely continuous. In this case the metric isomorphism is linear (sliding summation). It turns out that, under certain assumptions, a linear isomorphism is the only possible kind of metric one.

Theorem 5. *For n.d.s. whose spectrum is not of infinite multiplicity, the following kinds of isomorphism are equivalent: a) linear isomorphism, b) metric isomorphism, c) spectral isomorphism.*

The proof is based on a simple fact. An automorphism W of the unitary ring $\mathcal{L}$ is either linear or has infinite degree. (An automorphism has infinite degree if

$$WL^2_{F_0^1} \not\subset \sum_{k=1}^{n} \oplus L^2_{F_0^2(k)}, \quad n = 1, 2, ...).$$

Theorem 5 was also proved by Ya. G. Sinai (unpublished).

If the spectrum of an n.d.s. contains a component of countable multiplicity, then the problem of metric isomorphism becomes substantially more complicated. Examples show that the kinds of isomorphism indicated in Theorem 5 are, generally speaking, different. A. M. Kagan and L. V. Rykova independently proposed an example of n.d.s. that are spectrally isomorphic but have different entropy and hence are not metrically isomorphic (unpublished). At the same time one can give an example of metrically isomorphic n.d.s. for which the isomorphism between them cannot be linear. The first example of this kind is due to Ya. G. Sinai (unpublished).

The problem can be solved completely in the case of a homogeneous Lebesgue spectrum. Let us note that an n.d.s. has a homogeneous Lebesgue spectrum if and only if all elements of the spectral matrix of the n.d.s. $\|F_{ij}\|$ are absolutely continuous with respect to Lebesgue measure. In (^{8}) it is proved that all such n.d.s. are Kolmogorov automorphisms and have infinite entropy.

Theorem 6. *N.d.s. (of arbitrary dimensions) whose spectrum is homogeneous Lebesgue are metrically isomorphic to one another.*

The results of the present note can be transferred to the case of complex n.d.s. and to the case of continuous time.

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Note: Figure translations are in progress. See original paper for figures.