



Soviet-era science, translated into English

MATHEMATICS

I. L. BONDI

1962

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.45450>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

I. L. BONDI

ON ALMOST EVERYWHERE A -INTEGRABLE FUNCTIONS

(Presented by Academician P. S. Aleksandrov, 5 III 1962)

Definition 1. A measurable function $f(x)$ is called A -integrable on the interval $[a, b]$ if the following two conditions are satisfied:

1)

$$n \text{ mes}\{x : |f(x)| \geq n\} \rightarrow 0 \quad \text{as } n \rightarrow \infty; \quad (1)$$

2) there exists the finite limit

$$\lim_{n \rightarrow \infty} \int_a^b [f(x)]_n dx,$$

where

$$[f(x)]_n = \begin{cases} f(x), & \text{if } |f(x)| \leq n, \\ 0, & \text{if } |f(x)| > n. \end{cases}$$

This limit is called the A -integral of $f(x)$ over $[a, b]$ and is denoted by

$$(A) \int_a^b f(x) dx. \quad (2)$$

The idea of defining an integral by means of the limit (2) was used by Titchmarsh⁽⁶⁾. In the same work Titchmarsh showed that the integral defined as the limit (2) is not additive with respect to the integrand functions, and also that fulfillment of condition (1) ensures additivity. The definition of the integral as the limit (2), with condition (1) observed, was given by A. N. Kolmogorov in probabilistic form⁽³⁾, Ch. 6) and by J. C. Oxtoby from the point of view of the theory of functions⁽⁵⁾, Ch. 4). A number of works by P. L. Ulyanov⁽⁷⁻¹⁰⁾ are devoted to questions of application of the A -integral.

Since the existence of the A -integral of $f(x)$ on the interval $[a, b]$ does not imply the existence of the A -integral on an interval $[\alpha, \beta] \subset [a, b]$, the concept of an almost everywhere A -integrable function is introduced.

Definition 2. A function $f(x)$ is called **almost everywhere A -integrable** on the interval $[a, b]$ if:

- 1) $f(x)$ is A -integrable on $[a, b]$;
- 2) in the interval $[a, b]$ there is contained a set E of full measure such that $f(x)$ is A -integrable on every interval $[\alpha, \beta]$, the endpoints of which belong to the set E .

An important class of almost everywhere A -integrable functions is the class of functions conjugate to summable ones. A. N. Kolmogorov proved in work ⁽⁴⁾ that every function conjugate to a summable function satisfies condition (1), while Titchmarsh established that for it the limit (2) exists on the interval $[0, 2\pi]$, and also on almost every interval $[\alpha, \beta] \subset [0, 2\pi]$ ⁽⁶⁾.

Another important class of almost everywhere A -integrable functions is the class of functions representable by series of the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \quad (3)$$

or

$$\sum_{k=1}^{\infty} a_k \sin kx, \quad (4)$$

where $\{a_k\}$ is a sequence of numbers satisfying the conditions

$$a_k \rightarrow 0, \quad \sum_{k=1}^{\infty} |a_k - a_{k+1}| < \infty. \quad (5)$$

This was established by P. L. Ul'yanov in [7].

P. L. Ul'yanov proved that the functions of the classes mentioned above remain A -integrable after multiplication by "sufficiently good" functions. Namely, if $\varphi(x)$ and $\overline{\varphi}(x)$ are both bounded, and $f(x)$ is summable, then $\overline{f}(x)\varphi(x)$ is A -integrable (see [8]); or if $\varphi(x)$ and $\overline{\varphi}(x)$ are both of bounded variation, and $f(x)$ and $\overline{f}(x)$ are representable by series (3) and (4) under condition (5), then the products $f(x)\varphi(x)$ and $\overline{f}(x)\varphi(x)$ are A -integrable (see [7]).

However, arbitrary almost everywhere A -integrable functions may lose the property of A -integrability after multiplication even by "very good" functions. Namely, the following is true:

Theorem 1. For any function $\varphi(x)$ continuous on $[a, b]$ and not identical to a constant, one can find a function $f(x)$, almost everywhere A -integrable on $[a, b]$, such that the product $f(x)\varphi(x)$ will not be A -integrable on $[a, b]$.

In particular, not every function $f(x)$, almost everywhere A -integrable on $[a, b]$, has all A -integrals defined,

$$(A) \int_0^{2\pi} f(x) \cos kx dx; \quad (A) \int_0^{2\pi} f(x) \sin kx dx$$

$$(k = 1, 2, 3, \dots)$$

and, consequently, not every almost everywhere A -integrable function can be associated with its Fourier A -series.

In order to free the class of almost everywhere A -integrable functions from the noted defect, it is useful to introduce the following definition.

Definition 3. A function $f(x)$ is called **almost everywhere A -integrable in the narrow sense** on the interval $[a, b]$, if:

- 1) $f(x)$ is A -integrable on $[a, b]$;
- 2) there exists a point $x_0 \in [a, b]$ such that the sequence of functions $F_n(x)$, defined by the formulas

$$F_n(x) = \int_{x_0}^x [f(t)]_n dt,$$

converges almost everywhere on $[a, b]$ to some function $F(x)$, and the sequence of norms $F_n(x)$ is bounded in the space $L_p[a, b]$ for every $p \geq 1$.

Denote by E the set of points of convergence of the sequence $F_n(x)$. The measure of the set E is equal to $b - a$, and for any points α and β belonging to

sets E , there exists a finite

$$(A) \int_{\alpha}^{\beta} f(x) dx.$$

Thus, $f(x)$ is almost everywhere A -integrable also in the sense of Definition 2.

We shall call the function $F(x)$ an indefinite A -integral of $f(x)$. It is clear that the indefinite A -integral is defined only up to an additive constant, since any point of the set E may be taken as the point x_0 . It is easy to show that $F(x)$ belongs to the class $L_p[a, b]$ for every p .

We shall denote by $A^*[a, b]$ the class of all functions almost everywhere A -integrable in the narrow sense on the interval $[a, b]$. From Definition 3 it follows that $L[a, b] \subset A^*[a, b]$. Moreover, one can show that every function conjugate to a summable function, and also every function represented by a series (3) or (4) under condition (5), belongs to the class $A^*[0, 2\pi]$.

For functions of the class A^* the following theorems hold.

Theorem 2. Let $f(x) \in A^*[a, b]$; let $\varphi(x)$ be an absolutely continuous function whose derivative belongs to the space $L_p[a, b]$ for some $p > 1$; and let α and β be arbitrary points of the set E . Then the product $f(x)\varphi(x)$ is A -integrable on $[\alpha, \beta]$, and the formula for integration by parts holds:

$$(A) \int_{\alpha}^{\beta} f(x)\varphi(x) dx = F(\beta)\varphi(\beta) - F(\alpha)\varphi(\alpha) - \int_{\alpha}^{\beta} F(x)\varphi'(x) dx.$$

Theorem 2*. If $f(x) \in A^*[a, b]$, while $\varphi(x)$ is absolutely continuous, has period $b - a$, and $\varphi'(x)$ belongs to $L_p[a, b]$ for some $p > 1$, then the product $f(x)\varphi(x) \in A^*[a, b]$. If, in addition,

$$(A) \int_a^b f(x) dx = 0,$$

then the following formula holds:

$$(A) \int_a^b f(x)\varphi(x) dx = - \int_a^b F(x)\varphi'(x) dx.$$

Remark. The periodicity of the function $\varphi(x)$ is essential for the validity of Theorem 2*, since the product of a function conjugate to a summable function by the function $\varphi(x) = x$ may fail to be A -integrable on the interval $[0, 2\pi]$.

In particular, it follows from Theorem 2* that for every function of the class $A^*[0, 2\pi]$ the Fourier A -series is defined and coincides, up to an additive constant, with the differentiated Fourier series of some function belonging simultaneously to all classes L_p . On the other hand, relying on the results of I. A. Vinogradova (2), one can prove that every differentiated Fourier series of a function belonging to all classes $L_p[0, 2\pi]$ is the Fourier A -series of some function from $A^*[0, 2\pi]$. This follows from the following theorem.

Theorem 3. For every function $F(x) \in L_p[a, b]$ for all $p \geq 1$, there exists a function $f(x) \in A^*[a, b]$ such that the A -integral of $f(x)$ over $[a, b]$ is equal to zero and, for almost all $x \in [a, b]$,

$$F(x) = \int_a^x f(t) dt.$$

Relying on Theorem 3, it is easy to prove that an arbitrary measurable and almost everywhere finite function is the sum of some almost everywhere convergent on $[0, 2\pi]$ Fourier A -series of a function from $A^*[0, 2\pi]$, and consequently a Fourier A -series need not be summable by the Abel-Poisson method

to the function from which it is formed. However, if one assumes that the Fourier series A of some function $f(x) \in A^*[0, 2\pi]$ is summable almost everywhere on $[0, 2\pi]$ by the method A^* (see ⁽¹⁾, p. 30) to $f(x)$, then the trigonometric series conjugate to it will also be summable almost everywhere on $[0, 2\pi]$ by the method A^* to some function $\bar{f}(x)$ (see ⁽¹⁾, p. 605). The question arises whether the function $\bar{f}(x)$ must be A -integrable on $[0, 2\pi]$. This question must be answered in the negative, since one can construct a function $f(x)$ such that:

- a) $f(x) \in A^*[0, 2\pi]$;
- b) the Fourier series A of the function $f(x)$ is summable by the method A^* to $f(x)$ almost everywhere on $[0, 2\pi]$;
- c) the series conjugate to it is summable almost everywhere on $[0, 2\pi]$ by the method A^* to the function $\bar{f}(x)$, which for almost all $x \in [0, 2\pi]$ is expressed by the formula

$$\bar{f}(x) = -\frac{1}{\pi}(A) \int_0^\pi \frac{f(x+t) - f(x-t)}{2 \operatorname{tg} \frac{t}{2}} dt;$$

- d) the function $\bar{f}(x)$ is not A -integrable on $[0, 2\pi]$.

In conclusion let us consider the question of a change of variable in the A -integral.

Theorem 4. *Let the function $\varphi(t)$ be absolutely continuous and strictly monotone on $[\alpha, \beta]$, and map the interval $[\alpha, \beta]$ onto the interval $[a, b]$. For definiteness we shall assume that $\varphi(t)$ is increasing. Then, in order that the function $f[\varphi(t)]\varphi'(t)$ be A -integrable on the interval $[\alpha, \beta]$ and that the equality*

$$(A) \int_a^b f(x) dx = (A) \int_\alpha^\beta f[\varphi(t)]\varphi'(t) dt* \quad (6)$$

hold for every function $f(x)$, A -integrable on the interval $[a, b]$, it is necessary and sufficient that some set $E \subset [\alpha, \beta]$ of full measure can be represented as the union of sets E_1 and E_2 in such a way that for all $t \in E_1$

$$0 < m \leq \varphi'(t) \leq M < +\infty,$$

and for all $t \in E_2$

$$\varphi'(t) = 0.$$

Theorem 4 is a strengthening of the theorem of P. L. Ulyanov (¹⁰), since in it the restrictions imposed on the function $\varphi(t)$ are somewhat weakened. Theorem 4 also asserts that no further weakening of these restrictions (under the condition that $\varphi(t)$ is absolutely continuous and strictly monotone) is possible.

Let us also note that the conditions imposed in Theorem 4 on the function $\varphi(t)$ remain necessary for the validity of equality (6) also in the case where one considers only functions $f(x)$ belonging to the class $A^*[a, b]$.

Received
27 II 1962

CITED LITERATURE

- ¹ N. K. Bari, *Trigonometric Series*, Moscow, 1961.
- ² I. A. Vinogradova, DAN, **135**, No. 1, 9 (1960).
- ³ A. N. Kolmogorov, *Basic Concepts of Probability Theory*, Moscow-Leningrad, 1936.
- ⁴ A. Kolmogoroff, Fund. Math., **7**, 23 (1925).
- ⁵ Yu. S. Ochan, Mat. sbornik, **28** (70), 293 (1951).
- ⁶ E. Titchmarsh, Proc. Lond. Math. Soc., **29**, 49 (1929).
- ⁷ P. L. Ulyanov, Mat. sbornik, **35** (77), 469 (1954).
- ⁸ P. L. Ulyanov, Uch. zap. Mosk. univ., issue 181, Mathematics, **8**, 139 (1956).
- ⁹ P. L. Ulyanov, UMN, **11**, No. 5, 223 (1956).
- ¹⁰ P. L. Ulyanov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **60**, 262 (1961).

* We assume that $(\pm\infty) \cdot 0 = 0$.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.