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Abstract

Full Text

P. P. MOSOLOV

AN ANALOGUE OF THE FIRST BOUNDARY-VALUE PROBLEM FOR STABLE HYPOELLIPTIC OPERATORS

(Presented by Academician P. S. Aleksandrov on 25 XII 1961)

Let us consider the class of differential operators introduced in the work (^{1}). This class, defined in a bounded domain D of n-dimensional Euclidean space, is characterized by the fact that, if we write the differential operator $\mathcal{P}\left(x, \frac{\partial}{\partial x}\right)$ in the form

$$\mathcal{P}\left(x, \frac{\partial}{\partial x}\right) = \sum_{i_1 \ldots i_n} (-1)^{\sum_1^n i_k} \frac{\partial^{\sum_1^n i_k}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} a_{2i_1 \ldots 2i_n}(x) \frac{\partial^{\sum_1^n i_k}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} + \sum_{\alpha_{i_1} \ldots \alpha_{i_n}, s} (-1)^{\beta_1} D^{\beta_1}_{1, \alpha_{i_1} \ldots \alpha_{i_n}, s} c_{2\alpha_{i_1} \ldots 2\alpha_{i_s}, 2\alpha_{i_{s+1}}+1, \ldots, 2\alpha_{i_n}+1}(x) D^{\beta_2}_{2, \alpha_{i_1} \ldots \alpha_{i_n}, s}, \tag{1}$$

where

$$D^{\beta_1}_{1, \alpha_{i_1} \ldots \alpha_{i_n}, s} = \frac{\partial^{\beta_1}}{\partial x_{i_1}^{\alpha_{i_1}} \ldots \partial x_{i_s}^{\alpha_{i_s}} \ldots \partial x_{i_{s+k}}^{\alpha_{i_{s+k}}+1+k-2\left[\frac{k}{2}\right]} \ldots \partial x_{i_n}^{\alpha_{i_n}+1+n-s-2\left[\frac{n-s}{2}\right]}},$$

$$D^{\beta_2}_{2, \alpha_{i_1} \ldots \alpha_{i_n}, s} = \frac{\partial^{\beta_2}}{\partial x_{i_1}^{\alpha_{i_1}} \ldots \partial x_{i_s}^{\alpha_{i_s}} \ldots \partial x_{i_{s+k}}^{\alpha_{i_{s+k}}+2\left[\frac{k}{2}\right]-k} \ldots \partial x_{i_n}^{\alpha_{i_n}+2\left[\frac{n-s}{2}\right]-n+s}};$$

$a_{2i_1 \ldots 2i_n}(x) \geq k > 0$ for $x \in D$, $[k]$ is the least integer not less than k, then

$$\mathcal{H}(x, \sigma) + c \geq c_1 \sigma_1^{2m_1} \cdots \sigma_n^{2m_n}, \qquad c_1 > 0, \qquad \sigma_i^2 \geq 0,$$

$$c_1 \int_D \sum_{i_1 \ldots i_n} a_{2i_1 \ldots 2i_n}(x) \left[\frac{\partial^{\sum_1^n i_k} \varphi(x)}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}\right]^2 dx \geq$$

$$\geq \left| \int_D \left\{ \sum_{i_1 \dots i_n} a_{2i_1 \dots 2i_n}(x) \left[\frac{\partial^{\sum_1^n i_k} \varphi(k)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \right]^2 + \right. \right.$$

$$\left. \left. + \sum_{\alpha_{i_1} \dots \alpha_{i_n}, s} c_{2\alpha_{i_1} \dots 2\alpha_{i_n}+1}(x) D^{\beta_1}_{1, \alpha_{i_1} \dots \alpha_{i_n}, s} \varphi(x) \, D^{\beta_2}_{2, \alpha_{i_1} \dots \alpha_{i_n}, s} \varphi(x) \right\} dx \right| \geq$$

$$\geq c_2 \int_D \sum_{i_1 \dots i_n} a_{2i_1 \dots 2i_n}(x) \left[\frac{\partial^{\sum_1^n i_k} \varphi(x)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \right]^1 dx, \qquad \varphi(x) \in C_0^\infty(D). \tag{2}$$

and, moreover, there exist numbers $\alpha_i^0 > 0$, $i = 1, \dots, n$, such that for all α_i, $0 \leq \alpha_i \leq \alpha_i^0$, the inequality

$$c_1 \mathcal{H}(x, \sigma) + c_2 \geq \left| \sigma_1^{2m_1 - 2k_1 + 2\alpha_1} \cdots \sigma_n^{2m_n - 2k_n + 2\alpha_n} \right|,$$

$$m_i \geq k_i, \qquad \sum_1^n k_i \geq 1, \quad i = 1, \dots, n, \tag{3}$$

holds, where $\sigma_1^{2m_1} \cdots \sigma_n^{2m_n}$ is one of the terms of the polynomial $\mathcal{P}_1(x, \sigma)$,

$$\mathcal{P}_1(x, \sigma) = \mathcal{H}(x, \sigma) +$$

$$+ \sum_{\alpha_{i_1} \dots \alpha_{i_n}, s} \sigma_{i_1}^{2\alpha_{i_1}} \cdots \sigma_{i_n}^{2\alpha_{i_n}} \left[\sigma_{i_{s+2}}^2 \cdots \sigma_{i_{s+k}}^{2\left(1+k-2\left[\frac{k}{2}\right]\right)} \cdots \sigma_{i_n}^{2\left(1+n-s-2\left[\frac{n-s}{2}\right]\right)} + \right.$$

$$\left. + \sigma_{i_{s+1}}^2 \cdots \sigma_{i_{s+k}}^{2\left(2\left[\frac{k}{2}\right]-k\right)} \cdots \sigma_{i_n}^{2\left(2\left[\frac{n-s}{2}\right]-n+s\right)} \right],$$

$$\mathcal{H}(x, \sigma) = \sum_{i_1 \dots i_n} a_{2i_1 \dots 2i_n}(x) \, \sigma_1^{2i_1} \cdots \sigma_n^{2i_n}.$$

Consider the Hilbert space $H_H(D)$, which is the closure of the space $C_0^\infty(D)$ in the norm

$$\|\varphi\|_H = \left\{ \int_D \sum_{i_1 \dots i_n} a_{2i_1 \dots 2i_n}(x) \left| \frac{\partial^{\sum_1^n i_k} \varphi(x)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \right|^2 dx \right\}^{1/2}. \tag{4}$$

It is not difficult to see that $\|\varphi\|_H \geq \|\varphi\|_{L_2}$, i.e. $H_H(D) \subset L_2(D)$. If $f(x)$ belongs to $L_2(D)$, then $u_0(x)$ is called a generalized solution of the boundary-value problem if

$$\int_D \Bigg[\sum_{i_1 \dots i_n} a_{2i_1 \dots 2i_n}(x) \frac{\partial^{\sum_1^n i_k} u_0}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \frac{\partial^{\sum_1^n i_k} \varphi(x)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} +$$

$$+ \sum_{\alpha_{i_1} \dots \alpha_{i_n}, s} c_{2\alpha_{i_1} \dots 2\alpha_{i_n}+1}(x) D^{\beta_1}_{1,\alpha_{i_1} \dots \alpha_{i_n}, s} u_0 \, D^{\beta_2}_{2,\alpha_{i_1} \dots \alpha_{i_n}, s} \varphi \Bigg] \, dx =$$

$$= \int_D f(x)\varphi(x)\, dx, \qquad u_0(x) \in H_H(D) \tag{5}$$

for all $\varphi(x) \in H_H(D)$. Applying the scheme set forth in the work (2), and using condition (2), we obtain that there does indeed exist such a generalized solution of the boundary-value problem for any right-hand side from $L_2(D)$, and that this generalized solution is unique. The concept of solution thus introduced is a particular case of the concept of a strong generalized solution considered in (1), i.e. this generalized solution has inside the domain D sufficiently many square-summable derivatives, provided that the coefficients of the equation and the right-hand side are sufficiently smooth functions. Hence it follows directly that the generalized solution of the boundary-value problem $u_0(x)$ inside the domain D is an ordinary solution of the differential equation

$$\mathcal{P}\left(x, \frac{\partial}{\partial x}\right) u_0(x) = f(x).$$

It turns out that the functions $D^\alpha u_0(x)$ are smooth up to the boundary of the domain, provided that the coefficients of the equation, the right-hand side, and the boundary $\dot{D}$ of the domain are sufficiently smooth, and that the latter satisfies the following condition: in the coordinate system in which the hypoel-

the elliptic operator satisfies condition (2), the boundary of the domain must admit the representation

$$\sum_1^n f_i(x_i) = 0.$$

A point $x^0 \in \dot{D}$, where $\dot{D}$ is the boundary of the domain, is called ordinary if, at it, either

$$0 < \left| \prod_{i=1}^n \frac{\partial f_i(x_i^0)}{\partial x_i} \right| < \infty$$

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or, if some $\dfrac{\partial f_i(x_i^0)}{\partial x_i} = 0$, then $f_i'(x_i) \equiv 0$ for $x_i^0 - \varepsilon \leq x_i \leq x_i^0 + \varepsilon$. A neighborhood of an ordinary point $O(x^0)$ is called normal if the boundary points belonging to this neighborhood are ordinary and form a connected set. By virtue of the assumptions concerning the boundary of the domain, one can, by the transformation $x_i' = f_i(x_i)$ if $f_i'(x_i) \neq 0$, and $x_i' = x_i$ if $f_i'(x_i) \equiv 0$, map the part of the boundary $\dot{D}$ contained in the neighborhood $O(x^0)$ onto a part of the hyperplane

$$\sum_1^n \varepsilon_i x_i' = 0,$$

where ε_i is either 1 or 0. Obviously, after the indicated change of coordinates, the stably hypoelliptic operator will still satisfy conditions (2), (3). Denote the differential operator of order l obtained from the original one by the indicated substitution

$$\mathfrak{P}'\left(x', \frac{\partial}{\partial x'}\right) = \sum_{\alpha_1 \dots \alpha_n}' b_{\alpha_1 \dots \alpha_n}(x') \frac{\partial^l}{\partial x_1'^{\alpha_1} \dots \partial x_{n'}^{\alpha_n}}.$$

Here x^0, $u_0(x)$, $O(x^0)$, $f(x)$ pass respectively into $x^{0'}$, $u_0'(x')$, $O'(x^{0'})$, $f'(x')$. Let the vector

$$\eta = \{\beta_1^j, \dots, \beta_n^j\}, \qquad j = 1, \dots, n-1,$$

be parallel to the hyperplane

$$\sum_1^n \varepsilon_i x_i' = 0$$

and let

$$\mathcal{H}(x, \sigma) \geq c_j' \left| \sum_{i=1}^n \varepsilon_i \beta_i^j \right|^{2r_j} + c_j^2, \qquad c_j' > 0$$

for $|\sigma| \geq a$. Denote by

$$\mathcal{H}\left(1, \frac{\partial}{\partial x'}\right)$$

the operator obtained from

$$\mathcal{H}\left(x', \frac{\partial}{\partial x'}\right),$$

if in the latter $a_{2i_1 \dots 2i_n}(x') \equiv 1$.

Theorem 1. Let $b_{\alpha_1 \dots \alpha_n}(x')$ belong to the space

$$C^{[\frac{l}{2}]}(O'(x^{0'})) \cap C^{[\frac{l}{2}]+k}_{\sum_i \beta_i^j x_i'}(O'(x^{0'})),$$

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and let

$$D^{k-r_j}_{\sum_i \beta^j_i x'_i} f'(x')$$

belong to the space $L_2(O'(x^{0'}))$. Then

$$D^{k}_{\sum_i \beta^j_i x'_i} D^r u'_0(x')$$

belongs to the space $L_2(O'(x^{0'}))$, where D^{2r} is an arbitrary term of

$$\mathscr{H}\left(1, \frac{\partial}{\partial x'}\right).$$

Theorem 1 shows that, if the parameters of the equation (the coefficients and the right-hand side) are sufficiently smooth functions, then the solution at ordinary points of the boundary has sufficiently many derivatives in the tangential direction, uniformly bounded up to the boundary. The derivatives normal to the boundary can be expressed from the equation itself. Thus we arrive at the theorem:

Theorem 2. Let

$$\mathfrak{P}\left(x, \frac{\partial}{\partial x}\right) = \sum_{\alpha_1 \dots \alpha_n} a_{\alpha_1 \dots \alpha_n}(x) \frac{\partial^l}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

be a differential operator satisfying conditions (2), (3), let the point x^0 be ordinary, and let $O(x^0)$ be a normal neighborhood of the point x^0. Suppose

$$a_{\alpha_1 \dots \alpha_n}(x)$$

belongs to the space

$$C^{[\frac{l}{2}]+k}(O(x^0))$$

and the boundary $\dot{D}$ is such that $f_i(x_i)$ belongs to

$C^{3[\frac{l}{2}]+k}(O(x^0))$. Suppose, finally, that $D^{l+k} f(x)$ belongs to $L_2(O(x^0))$. Then $u_0(x)$ is a generalized solution of the boundary-value problem in a normal neighborhood of an ordinary point such that $D^i_\tau D^{l+j}_n u_0(x)$ belongs to $L_2(O(x^0))$, $i + j = k$, where D_τ is differentiation in an arbitrary tangential direction, and D_n is differentiation along the normal.

Consider the sets on the boundary of the domain D

$$x \in R^i_{n-1}, \qquad \text{if} \quad \frac{\partial f_i(x_i)}{\partial x_i} = 0, \quad x \in \bar{D},$$

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$$x \in R_{n-k}^{i_1 \dots i_k}, \qquad \text{if} \quad \frac{\partial f_{i_1}(x_{i_1})}{\partial x_{i_1}} = \dots = \frac{\partial f_{i_k}(x_{i_k})}{\partial x_{i_k}} = 0, \quad x \in \bar{D}.$$

We shall use the theorem on the possible growth of a strong generalized solution as it approaches the boundary of the domain D (^{1}). Using this theorem and the statements formulated above, we arrive at the following theorem.

Theorem 3. If x^0 is a point of the set

$$\bar{D} \Big/ \bigcup_{i_1 \dots i_{k+1}} R_{n-k-1}^{i_1 \dots i_{k+1}},$$

$O(x^0)$ is a neighborhood of the point x^0, and

$$O(x^0) \cap \bar{D} \subset \bar{D} \Big/ \bigcup_{i_1 \dots i_{k+1}} R_{n-k-1}^{i_1 \dots i_{k+1}}.$$

If the coefficients of the equation belong to

$$C^{\left[\frac{l}{2}\right]+2l+k}(O(x^0)),$$

the right-hand side $f(x)$ belongs to $W_2^{(2l+k)}(O(x^0))$, and, moreover, in $O(x^0)$

$$\left|f_{i_s}^{(j)}(x_{i_s})\right| < r(x_{i_s}, N_{i_s})^{\frac{\left[\frac{l}{2}\right]+k+2l-2}{\min_i \alpha_i^0} - l + \left[\frac{l}{2}\right] - 1}, \qquad 1 \le j \le l+k,$$

where N_{i_s} is the set of zeros of the function $f'_{i_s}(x_{i_s})$, then $D_\tau^i D_n^{l+j} u_0(x)$ belongs to $L_2(O(x^0))$, $i + j = k$.

The constructed generalized solution of the boundary-value problem satisfies the following boundary conditions at ordinary points of the boundary (if it is, of course, sufficiently smooth):

$$D^\alpha u_0\big|_\Gamma = 0,$$

where D^α is such that there exists a derivative D^β, $\beta \neq 0$, and $[D^\alpha D^\beta]^2$ is a term of the operator

$$\mathcal{H}\left(1, \frac{\partial}{\partial x}\right).$$

The boundary conditions extend by continuity to the sets $R_{n-k}^{i_1 \dots i_k}$, if the solution is differentiable sufficiently many times in the domain $\bar{D}$.

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Note: Figure translations are in progress. See original paper for figures.

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