MATHEMATICS

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Abstract

Full Text

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ON CONDITIONAL STABILITY IN CRITICAL CASES

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Consider a system of differential equations of the form

$$\frac{dx}{dt} = Px + X(x, y), \qquad \frac{dy}{dt} = Qy + Y(x, y), \tag{1}$$

where $x = \{x_1, \dots, x_n\}$ and $X = \{X_1, \dots, X_n\}$ are n-dimensional vectors; $y = \{y_1, \dots, y_k\}$ and $Y = \{Y_1, \dots, Y_k\}$ are k-dimensional vectors; $P = \{p_{ij}\}$ is a constant square matrix of order n, all of whose eigenvalues have positive real parts; $Q = \{q_{ij}\}$ is a constant square matrix of order k, all of whose eigenvalues have nonpositive real parts; the functions X and Y are power series in x_i, y_i, beginning with terms of degree not lower than two.

It is known ([1–3]) that in this case there exists a k-dimensional analytic vector function $g(x)$ such that $g(0) = 0$; every solution of system (1) beginning at $t = 0$ on the manifold

$$y = g(x) \tag{2}$$

in a sufficiently small neighborhood of the origin remains on this manifold for all $t \leq 0$, and the zero solution of system (1) is asymptotically stable as $t \to -\infty$, if the initial data of the solutions satisfy condition (2), i.e., in this case conditional asymptotic stability as $t \to -\infty$ occurs.

We shall show that, under certain additional assumptions, conditional asymptotic stability as $t \to +\infty$ also occurs. Alongside system (1), consider the "truncated" system

$$\frac{dy}{dt} = Qy + Y(0, y). \tag{3}$$

Theorem 1. *If the zero solution of system (3) is asymptotically stable as $t \to +\infty$ independently of the form of the terms of order higher than N ([4]), and if the expansion of the function $X(0, y)$ begins with terms of order not lower than $N + 1$, then there exists an n-dimensional continuously differentiable vector* *function $f(y)$ such that $f(0)=0$; every solution of system (1) beginning at $t=0$ on the manifold*

$$x = f(y) \tag{4}$$

in a sufficiently small neighborhood of the origin remains on this manifold for all $t \geq 0$, and the zero solution of system (1) is asymptotically stable as $t \to +\infty$, if the initial data satisfy condition (4).

The manifold (4) may be obtained in the following way. Through all points of the k-dimensional ball $x=0$, $\|y\| < c$, where c is a sufficiently small positive number, pass the solutions of system (1). Denote by S_t the surface formed by the points of these solutions corresponding to the time t. The manifold (4) is the topological limit of the surface S_t as $t \to -\infty$.

Regarding the behavior of solutions not lying on the manifolds (2) and (4), one can make the following assertion:

Theorem 2. *Suppose the conditions of Theorem 1 are satisfied. Then there exists a neighborhood U of the origin such that every solution not lying on the manifold (2) leaves U as time decreases; every solution not lying on the manifold (4) leaves U as time increases.*

If system (1) does not satisfy the conditions of Theorem 1, then it can sometimes be brought to such a form that it will satisfy these conditions, by means of the following change of variables.

Consider the system of partial differential equations[4]

$$\sum_{i=1}^{k} \frac{\partial u_s}{\partial y_i}(q_{i1}y_1 + \dots + q_{ik}y_k) + Y_i(u_1, \dots, u_n, y_1, \dots, y_k) =$$

$$= p_{s1}u_1 + \dots + p_{sn}u_n + X_s(u_1, \dots, u_n, y_1, \dots, y_k) \qquad (s = 1, \dots, n). \tag{5}$$

This system can always be satisfied by formal series of the form

$$u_s = u_s^{(2)}(y_1, \dots, y_k) + u_s^{(3)}(y_1, \dots, y_k) + \cdots, \tag{6}$$

where $u_s^{(l)}$ are forms of degree l in the variables $y_1, \dots, y_k$. Make in system (1) the change of variables

$$x_s = \xi_s + u_s^{(2)}(y_1, \dots, y_k) + \dots + u_s^{(N)}(y_1, \dots, y_k);$$

then it is not difficult to verify that the resulting system will again be of the form (1), but the expansion of the function $X(0, y)$ will begin with terms of order no lower than $N+1$. It may turn out that, for sufficiently large N, the zero solution of the corresponding "shortened" system is asymptotically stable independently of the form of the terms of order higher than N, i.e., the conditions of Theorem 1 will be satisfied.

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CITED LITERATURE

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[2] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, Moscow, 1958.
[3] S. Lefschetz, *Geometrical Theory of Differential Equations*, IL, 1961.
[4] I. G. Malkin, *Theory of Stability of Motion*, Moscow–Leningrad, 1952.

Note: Figure translations are in progress. See original paper for figures.