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Abstract

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HYDROMECHANICS

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ON THE STABILITY OF THE FLOW OF A HIGHLY CONDUCTING LIQUID ACROSS A MAGNETIC FIELD

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1. The flow of a conducting incompressible viscous liquid between plane-parallel plates across an external uniform field, frozen outside the liquid into an ideal conductor, was investigated by Hartmann ([1]). The exact solution of the stationary problem obtained by him depends on the dimensionless parameter

$$M = \frac{lH_0}{c}\sqrt{\frac{\sigma}{\eta}} \tag{1}$$

(here l is the half-width of the channel, H_0 the external field, σ and η the conductivity and dynamic viscosity of the liquid, and c the speed of light).

If $M \ll 1$, the flow becomes Poiseuille flow. If, however, $M \gg 1$, then the flow velocity is constant along the transverse section of the channel everywhere, with the exception of thin boundary layers of thickness $\sim l/M$, in which the velocity falls to zero. The magnitude of the velocity is determined by the relation

$$V_0 = -\frac{dP}{dx}\frac{lc}{H_0\sqrt{\sigma\eta}}, \tag{2}$$

where dP/dx is the constant pressure gradient in the direction of motion.

The magnetic field H_x, oriented along the velocity, for $M \gg 1$ varies over the cross-section as follows:

$$H_x = H_0\lambda z, \tag{3}$$

where the coordinate z is measured along $\mathbf{H}_0$ from the plane of symmetry, and $\lambda = \frac{4\pi}{H_0^2}\frac{dP}{dx}$ is a constant of dimension inverse length. Relation (3) is violated only in the boundary layers, where H_x falls to 0.

Let us note that if one passes to a coordinate system moving with velocity V_0, the limiting solution under consideration coincides with the solution of the problem of equilibrium of an ideally conducting liquid with the same geometry in a gravitational field

$$g = -\frac{1}{\rho}\frac{dP}{dx}.$$

The lines of force of the magnetic field in the liquid have the form of parabolas with sagging depth $L = l^2\lambda/2$.

The influence of a change in the velocity profile on the onset of ordinary turbulence (the winding-up of a vortex by the velocity gradient) was investigated by Lock ([2]). It is also of interest to consider the influence of the configuration of the magnetic field on stability.

2. We shall assume that $M \gg 1$, and $V_0 \ll c$. Then, in a coordinate system moving with the liquid, $H_z = H_0$, and H_x is given by relation (3). We approximate the boundary layers by surface currents, in connection with which we shall abandon the requirement that the tangential components of the perturbations of velocity and field vanish at the boundaries. We shall also assume that the quantities ν and $c^2/4\pi\sigma$ are sufficiently small, and shall neglect dissipation in nonstationary processes.

Let a small perturbation of the velocity $\mathbf{v}$ and of the magnetic field $\mathbf{h}$ arise in the fluid. Linearizing the equations of magnetohydrodynamics, we obtain:

$$\frac{\partial \mathbf{h}}{\partial t} + v_z \frac{d\overrightarrow{\mathcal{H}}}{dz} = (\overrightarrow{\mathcal{H}}\nabla)\mathbf{v}, \tag{4}$$

$$\frac{\partial}{\partial t}\operatorname{rot}\mathbf{v} = \frac{1}{4\pi\rho}\operatorname{rot}\left\{(\overrightarrow{\mathcal{H}}\nabla)\mathbf{h} + h_z\frac{d\overrightarrow{\mathcal{H}}}{dz}\right\}. \tag{5}$$

(here $\mathcal{H}_x = H_0\lambda z,\ \mathcal{H}_y = 0,\ \mathcal{H}_z = H_0$.)

In addition to (4), we have:

$$\operatorname{div}\mathbf{h} = 0, \qquad \operatorname{div}\mathbf{v} = 0. \tag{6}$$

For $z = \pm l$, h_z and v_z are equal to zero. With the aid of (4), from this one can obtain the conditions for v_z:

$$v_z(\pm l) = 0, \qquad \frac{\partial v_z(\pm l)}{\partial z} = 0. \tag{7}$$

Eliminating $\mathbf{h}$ from the equations, we have

$$\frac{\partial^2}{\partial t^2}\operatorname{rot}\mathbf{v} = \frac{1}{4\pi\rho}\operatorname{rot}\{(\overrightarrow{\mathcal{H}}\nabla)\mathbf{v}\}. \tag{8}$$

Consider perturbations lying in the xz plane and independent of y. We seek a solution in the form

$$\mathbf{v} = \mathbf{v}(z)e^{i(\omega t + kx)}. \tag{9}$$

Eliminating v_x, we obtain the following equation for $v_z \equiv v$:

$$\frac{d}{dz}\left\{\omega^2 + \frac{H_0^2}{4\pi\rho}\left(2ik\lambda z + \frac{d}{dz}\right)^2\right\}\frac{dv}{dz} = k^2\left\{\omega^2 + \frac{H_0^2}{4\pi\rho}\left(2ik\lambda z + \frac{d}{dz}\right)^2\right\}v. \tag{10}$$

Introduce the dimensionless coordinate $s = z/l$ and the parameters

$$\varkappa = l^2 k\lambda = -\frac{4\pi}{H_0^2}\frac{dP}{dx}kl^2, \qquad \Lambda = \frac{l\omega\sqrt{4\pi\rho}}{H_0}.$$

Equation (10) takes the form

$$\frac{d}{ds}\left\{\Lambda^2 + \left(2i\varkappa s + \frac{d}{ds}\right)^2\right\}\frac{dv}{ds} = (kl)^2\left\{\Lambda^2 + \left(2i\varkappa s + \frac{d}{ds}\right)^2\right\}v. \tag{10'}$$

A complete investigation of equation (10′) is associated with computational difficulties. Therefore we shall restrict ourselves to considering one of the particular cases, which is of independent interest for some problems of cosmic electrodynamics.

3. If long waves ($kl \ll 1$) are considered, the right-hand side of equation (10′) may be neglected, since it is proportional to the small parameter at the lower derivative. We shall assume, however, that the field is so weak that $L \gg l$ and that such wavelengths are possible for which, simultaneously, $\varkappa \gg 1$ (this requirement can be satisfied for large dP/dx without violating the other conditions).

Neglecting the right-hand side in (10′) and taking into account the operator identity

$$\left(2i\varkappa s + \frac{d}{ds}\right)f = e^{-i\varkappa s^2}\frac{d}{ds}e^{i\varkappa s^2}f, \tag{11}$$

we have

$$\frac{d}{ds}\left\{\Lambda^2 + e^{-i\varkappa s^2}\frac{d^2}{ds^2}e^{i\varkappa s^2}\right\}\frac{dv}{dx} = 0. \tag{12}$$

One of the solutions, $v = \text{const}$, is obvious, and the others are found by quadratures. Two solutions are even, and two are odd:

$$v_1 = 1; \qquad v_2 = \int_0^s \sin\Lambda s'\, e^{-i\varkappa s'^2}\, ds';$$

$$v_3 = \int_0^s \cos\Lambda s'\, e^{-i\varkappa s'^2}\, ds'; \qquad v_4 = \int_0^s \int_0^{s'} e^{-i\varkappa(s'^2 - s''^2)} \sin\Lambda(s'' - s')\, ds''\, ds'. \tag{13}$$

For even solutions the characteristic equation is trivial: $\sin\Lambda = 0$. Hence $\Lambda_n = n\pi$. All solutions are stable.

After simple but cumbersome transformations we obtain, in the case of odd solutions,

$$\frac{\operatorname{tg}\Lambda}{\Lambda} = \frac{1}{\varkappa\Lambda}\frac{\displaystyle\int_0^1 (\sin\varkappa\cos\varkappa x^2 - \cos\varkappa\sin\varkappa x^2)\,\frac{\sin\Lambda\cos\Lambda x - x\cos\Lambda\sin\Lambda x}{1 - x^2}\,dx}{\left(\displaystyle\int_0^1 \cos\varkappa x^2\cos\Lambda x\,dx\right)^2 + \left(\displaystyle\int_0^1 \sin\varkappa x^2\cos\Lambda x\,dx\right)^2}. \tag{14}$$

The function on the right-hand side is bounded, and therefore its graph intersects all branches of the curve $\operatorname{tg}\Lambda/\Lambda$, with the possible exception of the branch lying between $-\pi/2$ and $\pi/2$. Consequently, there exists an infinite set of stable solutions. Instability can be associated only with the indicated branch.

Let us consider wavelengths $\varkappa \gg 1$. Taking the slowly varying functions out from under the integrals and replacing their values by those at zero, we integrate the rapidly oscillating functions from 0 to ∞. Putting $\gamma = i\Lambda$, we obtain

$$\operatorname{ch}\gamma = \frac{\sqrt{\pi\varkappa}}{\sin(\varkappa - \pi/4)}, \tag{15}$$

whence (assuming that γ is sufficiently large and $\operatorname{ch}\gamma \simeq e^{|\gamma|}/2$)

$$\gamma = \pm\ln\frac{\sin(\varkappa - \pi/4)}{2\sqrt{\varkappa\pi}}. \tag{16}$$

For $\varkappa \to \pi/4 + n\pi$ the requirement of slow variation of the terms containing Λ is not satisfied. It is clear, however, that as $\varkappa$ tends to $\pi/4 + n\pi$, a resonant increase of the increment occurs. If $\sin(\varkappa - \pi/4) > 0$, the instability has the character of standing waves; but if $\sin(\varkappa - \pi/4) < 0$, traveling waves are excited. The instability grows most intensively if an integer plus one quarter number of waves fits along the length L.

The increment, expressed in dimensional quantities, has the form

$$\frac{1}{\tau} = \frac{H_0}{l\sqrt{4\pi\rho}} \ln\left\{ \frac{H_0}{4\pi l\sqrt{k\,dP/dx}} \sin\left(\frac{4\pi}{H_0^2}\frac{dP}{dx}\,kl^2 - \frac{\pi}{4} \right) \right\}. \tag{17}$$

From (17) it is seen that ν and σ play no role if (besides $M \gg 1$) the inequality $lH_0 \gg \sqrt{4\pi\rho}\,\nu$ is satisfied.

When the instability develops, in some regions the lines of force close on themselves, while in others the transverse field is amplified. One may say that the external field is expelled, becoming concentrated in thin layers. Such a configuration, in turn, is unstable; moreover, both types of instability considered by Kruskal and Schwarzschild ([4]) occur simultaneously—the instability in a gravitational field and the instability of a current-carrying cord.

The development of these instabilities must lead to the formation of clumps with closed lines of force, capable of moving freely in space.

4. Above we considered the case in which the density of potential (and, consequently, kinetic) energy is much greater than the density of the magnetic energy of the external field. Phenomena are associated with similar circumstances in which an initially weak field is amplified as a result of the stretching of the lines of force during the motion of conducting masses.

The instability investigated in the preceding paragraphs restricts the class of flows in which magnetohydrodynamic self-excitation of regular magnetic fields is possible, since cases must be excluded in which the lines of force in certain regions have the form of strongly elongated parabolas.

Let us briefly dwell on the problem of turbulent self-excitation of the field. On this question there exist two different points of view. Batchelor [(4)] considers that turbulent pulsations destroy the correlation of the magnetic field, and that it is therefore associated with the minimum scale of turbulence and correspondingly possesses a small energy. Other authors (see, for example, [(5,6)]) believe that the correlation of the field is preserved, and that in a state of dynamic equilibrium the energy of the field is of the order of the energy of the principal scale of the flow. In accordance with what has been set forth, Batchelor’ s point of view is preferable, since there exists a real mechanism for the destruction of the correlation of the field. It seems probable that this mechanism is not the only one and that in other cases, under strong deformations of the lines of force, an analogous instability occurs.

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Note: Figure translations are in progress. See original paper for figures.