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Abstract

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MATHEMATICS

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ON THE THEORY OF ALGEBRAIC LINEAR GROUPS

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Algebraic linear groups have been studied by E. Kolchin ([1,2]), K. Chevalley ([3,4]) and A. Borel ([5]). These authors studied nilpotent and solvable connected algebraic groups mainly over algebraically closed fields. Thus, in ([5]) the structure of connected nilpotent algebraic groups over an algebraically closed perfect field was established.

The main purpose of the present article is the study of arbitrary nilpotent algebraic groups over a perfect field. The terminology of the article largely coincides with the terminology adopted in ([3,5]).

By an algebraic linear group over a field P we mean a group closed in the Zariski topology over P. All topological concepts used below always refer to the Zariski topology.

The following assertions are of some independent interest.

Theorem 1. *Every periodic algebraic group of matrices Γ is finite (the field has characteristic zero).*

Proof. The field P may be assumed algebraically closed and $\Gamma \subset GL(n, P)$. If K is the connected component in Γ, then Γ/K is finite. Since every periodic group of matrices, by Schur' s theorem, has an abelian normal divisor of finite index, and its closure in the Zariski topology is abelian (([8], Ch. 8), every connected periodic group of matrices is abelian. Hence K is abelian, and, in view of the complete reducibility of the periodic group K, it is a diagonal periodic group. But the irreducible components of an algebraic group are algebraic (([1], Ch. I). From this the finiteness of K, and therefore also the finiteness of Γ, follows easily.

For an arbitrary linear group Γ, by Γ^* we shall denote the smallest algebraic group containing Γ. It is known (([4], Ch. 3) that Γ^* is a nilpotent group if Γ is nilpotent.

We give a simple proof of this assertion.

Theorem 2. *The closure of a nilpotent group Γ in the Zariski topology is nilpotent.*

Proof. Let Γ be a group of nilpotency class k. Then the compound commutator

$$[g_1, g_2, \dots, g_{k+1}] = e,$$

where $g_1, g_2, \dots, g_{k+1} \subset \Gamma$, e is the identity of Γ. Consider the mapping $g \to [g_1, \dots, g_k, g]$ for fixed $g_1, g_2, \dots, g_k \subset \Gamma$. By the continuity of this mapping, the preimage of the identity element is closed and contains Γ. The intersection of the preimages of the identity element over all possible $g_1, \dots, g_k \subset \Gamma$ will be the smallest algebraic nilpotent group containing Γ.

A nilpotent algebraic group is called topologically nilpotent if it possesses a central series consisting of algebraic subgroups. Since the centralizer of an algebraic group is an algebraic group, every algebraic nilpotent group is topologically nilpotent.

Let us note that the closure of a locally nilpotent but not nilpotent linear group in the Zariski topology is not locally nilpotent.

If Γ is an irreducible nilpotent linear group, then the index of its center Z is finite (([6], Ch. 3).

It follows that:

Theorem 3. *An irreducible nilpotent linear group is algebraic if and only if its center is algebraic.*

Definitions. We shall call a matrix $a \in GL(n, P)$ a d-matrix (or semisimple) if over some extension Σ of the field P there exists a diagonal matrix similar to a.

We shall call a matrix $u \in GL(n, P)$ unipotent if all its eigenvalues are equal to one.

A group A (respectively B) in $GL(n, P)$, every matrix of which is a d-matrix (respectively unipotent), will be called a d-group (respectively a unipotent group).

Let K be an algebraically closed perfect field; then every connected algebraic nilpotent group $\Gamma \subset GL(n, K)$ is represented in the form of a direct product $\Gamma = \Gamma_s \cdot \Gamma_u$, where Γ_s is the algebraic connected central subgroup of all d-matrices, and Γ_u is the subgroup of all unipotent matrices ([5]).

It turns out that a more general theorem is true:

Theorem 4. *Every algebraic nilpotent group Γ over an arbitrary perfect field P is represented in the form of a direct product $\Gamma = \Gamma_s \cdot \Gamma_u$, where Γ_s is the algebraic subgroup of all d-matrices, and Γ_u is the subgroup of all unipotent matrices from Γ.*

Proof. As shown in (^{7}), every nilpotent linear group Γ over a perfect field P is contained in a direct product $F \cdot H$, where F is a nilpotent d-group and H is a unipotent group. It is proved there also that a locally nilpotent group of matrices is completely reducible if and only if it is a group of d-matrices.

Every element $g \in GL(n, P)$ is represented uniquely in the form of a product $g = au$, where a is a d-matrix, u is a unipotent matrix, and $au = ua$. If g is contained in the algebraic group Γ, then a and u also belong to Γ (^{3}), Ch. 2). Hence, if Γ is a nilpotent algebraic group, then every element $g \in \Gamma$ is represented uniquely in the form $g = g_{sg}u$, where g_s is a d-matrix, g_u is a unipotent matrix, and $g_{sg}u = g_{ug}s$, with $g_s, g_u \in \Gamma$.

The set of all d-matrices in Γ forms a subgroup Γ_s, and the set of all unipotent matrices a subgroup Γ_u of the group Γ (see (^{7})). Obviously, Γ_s and Γ_u are normal divisors of Γ and $\Gamma = \Gamma_s \cdot \Gamma_u$ is a direct product. Γ_s is a completely reducible group, being a homomorphic image of Γ; hence Γ_s is algebraic. The theorem is proved.

Since every solvable irreducible group has an abelian normal divisor of finite index $((^6), Ch.I)$, it follows that

Lemma 1. *A connected irreducible solvable group of matrices is commutative.*

The following proposition is a direct extension of Borel' s theorem to the case of an arbitrary perfect field.

Theorem 5. *If Γ is a connected nilpotent algebraic group, then Γ is represented in the form of a direct product $\Gamma = \Gamma_s \cdot \Gamma_u$, where Γ_s is the algebraic connected central d-subgroup of Γ, and Γ_u is the subgroup of all unipotent matrices.*

The proof follows from Theorem 4 and Lemma 1 and from the connectedness of the irreducible image of a connected group.

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