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1962

SovietRxiv

Abstract

Full Text

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ON A SEMIGROUP OF OPERATORS IN A FRÉCHET SPACE

(Presented by Academician V. I. Smirnov on 30 X 1961)

§ 1. Introduction

Some Fréchet spaces (F-spaces) are beginning to play an important role in the study of partial differential equations. For example, such a space is the class of continuous functions on a locally compact space with the usual metric, or the class of infinitely differentiable functions on a compact space, if, in metrizing it, we require that certain differential operators be continuous in this space ([2,5]).

Let us note one of the differences between Banach spaces and Fréchet spaces*. Let $\{T(t);\, t \geqslant 0\}$ be a semigroup of operators in a Banach space, strongly continuous at $t = 0$. Then there exists an $\alpha > 0$ such that the semigroup**

$$T^0(t) = e^{-\alpha t} T(t)$$

satisfies the condition (see ([2])):

$$\text{The set } \{T^0(t)x;\, t \geqslant 0\} \text{ is bounded for every } x \in X. \qquad (*)$$

In Fréchet spaces this is, generally speaking, not so; an example is the translation semigroup: $T(t)x(s) = x(s+t)$ in the class of continuous functions on $(-\infty, \infty)$. (The class of continuous functions on $(-\infty, \infty)$ is a Fréchet space with seminorms ([5]) $\|x\|_k = \sup_{|s| \leqslant k} |x(s)|$ $(k = 1, 2, \ldots)$.)

The fundamental Hille–Yosida–Phillips theorem (([1]), $p.624$) remains valid for a semigroup in an F-space if condition $(*)$ is satisfied for every x, but it loses its force if $(*)$ is satisfied only on some everywhere dense subset of the space. The theorems obtained here are, in their nature, close to the results of Feller ([2]), which belong to another branch of semigroup theory.

The purpose of the present paper is to generalize the Hille–Yosida–Phillips theorem to those semigroups in an F-space for which condition $(*)$ is satisfied only on some everywhere dense subset. As an application, in § 4 we shall consider the question of extending a strongly continuous semigroup from a Banach space to a Fréchet space containing it as a subspace.

§ 2. Preliminary lemmas and definitions

I. A **Fréchet space** is a locally convex metrizable complete space.

II. A **strongly continuous semigroup of operators** is a family $\{T(t);\, t \geqslant 0\}$ of bounded operators mapping the F-space X into itself and satisfying the following con–

* The definitions will be given in § 2.
** In what follows we shall everywhere replace $T(t)$ by the equivalent semigroup $T^0(t)$.

conditions:

A. $T(t+s) = T(t)T(s),\ s, t \geqslant 0$.
B. $T(0) = E$ (E is the identity operator).
C. $\lim_{t\to 0} T(t)x = x$ for all $x \in X$.

III. The principle of uniform boundedness carries over to Fréchet spaces: let $\{T(t),\, t \geqslant 0\}$ be a family of bounded operators in X such that the set $\{T(t)x;\, t \geqslant 0\}$ is bounded for each $x \in X$. Then the relation

$$\lim_{x\to 0} T(t)x = \theta$$

holds uniformly in t (θ is the zero element of the space).

IV. Consider the subsets of X

$$\Sigma = \{x : \sup_{t\geqslant 0} \|T(t)x\|_k < \infty, \quad k = 1, 2, \ldots\}, \tag{1}$$

$$S = \{x : \sup_k \|x\|_k < \infty\} \tag{2}$$

with seminorms, respectively,

$$P_k(x) = \sup_{t\geqslant 0} \|T(t)x\|_k \quad (k = 1, 2, \ldots), \tag{3}$$

$$= \sup_k \|x\|_k. \tag{4}$$

As a result of the topology generated by these seminorms, Σ is an F-space* and S is a B-space.

V. S, as a subset of X, is everywhere dense in X.

§ 3. **Generalization of the Hille–Yosida–Phillips theorem to Fréchet spaces.** First of all we give a sufficient condition under which the semigroup $T(t)$ satisfies condition (*) on an everywhere dense subset of the space X.

Theorem 1. If $T(t)$ is a strongly continuous semigroup of operators and

$$T(t)S \subseteq S, \tag{5}$$

then $S \subseteq \Sigma$ and, consequently, (*) is fulfilled on an everywhere dense subset of the space X.

Proof is based on § 2, IV and V.

Example. In the case of the translation semigroup mentioned in § 1, the Banach space S is $C(-\infty, \infty)$, and condition (5) is, obviously, fulfilled.

In what follows we shall consider only such semigroups $\{T(t); t \geqslant 0\}$ for which condition (*) is satisfied on an everywhere dense set Σ. First of all, it is easy to see (§ 2, IV) that $T(t)$ is strongly continuous also in Σ, and thus (5) holds:

Theorem 2. Let the semigroup $\{T(t);\, t \geqslant 0\}$ in Σ satisfy conditions (*), and also A, B, C of § 2, II.

The infinitesimal generating operator** A is closed in Σ, and its domain of definition $D(A)$ is everywhere dense in Σ.

For every $\lambda > 0$ there exists R_λ–a bounded linear operator mapping Σ into itself–and

$$(\lambda E - A)R_\lambda x = x, \quad \text{if } x \in \Sigma;$$

$$R_\lambda(\lambda E - A)x = x, \quad \text{if } x \in D(A).$$

* The topology in a Fréchet space can be specified by means of a countable number of seminorms ((4), p. 208). We shall denote these seminorms by $\| \dots \|_k$.

** On the concept of an infinitesimal generating operator see (1), p. 614 or (5). We everywhere regard A as an operator in Σ, even if it can be extended to a broader domain.

For every $x \in \Sigma$ the set

$$\{\lambda^n R_\lambda^n x;\ \lambda > 0,\ n = 0, 1, 2, \dots\} \tag{6}$$

is bounded in Σ.

For every $x \in \Sigma$

$$T(t)x = \lim_{\lambda\to\infty} e^{-\lambda t} \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} (\lambda R_\lambda)^k x. \tag{7}$$

It is known that the boundedness of the set (6) is necessary and sufficient for the operator A to generate a strongly continuous semigroup in a Banach space. This is the content of the Hille–Yosida–Phillips theorem. The necessary conditions in a Fréchet space are as follows:

Theorem 3. Let $\{T(t);\, t \geq 0\}$ be a strongly continuous semigroup of operators in X, and let condition $(*)$ be satisfied on an everywhere dense subset $\Sigma \subset X$. Then:

I. If $x \in \Sigma$, then

$$\sup_{\lambda>0,\ n\geq 0} \|\lambda^n R_\lambda^n x\|_k = \sup_{t\geq 0} \|T(t)x\|_k.$$

II. If K is a bounded set in X and

$$\sup_{\tau\leq t,\ x\in K} \|T(\tau)x\|_k = v_k(t),$$

then for every $\varepsilon > 0$ and $x \in \Sigma \cap K$

$$\|\lambda^n R_\lambda^n x\|_k \leq v_k(N) + \varepsilon, \quad \text{provided that } \lambda N > n \text{ and } \lambda > \lambda_0(x, \varepsilon). \tag{8}$$

Proof is based on (6), on the inequality

$$\frac{\lambda^n}{(n-1)!} \int_N^\infty e^{-\lambda s} s^{n-1}\, ds \leq \frac{\lambda N}{\inf_{i<n} |i - \lambda N|^2}$$

and on the well-known relation between $R_\lambda x$ and $T(t)x$ for $x \in \Sigma$ [(5)].

Condition (8) is not only necessary, but in a certain sense also sufficient. Namely, the following holds:

Theorem 4. Let:

1. R_λ be a linear, but not necessarily bounded, operator in X for every $\lambda > 0$.
2. The set

$$\left\{ x : \sup_{\lambda>0,\ n\geq 0} \|\lambda^n R_\lambda^n x\|_k < \infty \right\} \tag{9}$$

be everywhere dense in X.

3. The set (9) be a Fréchet space with seminorms

$$\pi_k(x) = \sup_{\lambda>0,\ n\geq 0} \|\lambda^n R_\lambda^n x\|_k$$

(we denote this space by Σ^0).

4. R_λ satisfy condition (8).

Then:

1′. The set (6) is bounded, and R_λ is a bounded operator in Σ^0 for every $\lambda > 0$.

2′. A generates a strongly continuous semigroup of operators $T(t)$ in X such that

$$\|T(t)x\|_k < v_k(N),$$

if $t < N$ and $x \in K$.

3′. Σ^0 contains all and only those elements for which

$$\sup_{t\geq 0} \|T(t)x\|_k < \infty.$$

4°. The seminorms in Σ^0 are

$$\mathfrak{p}_k(x) = \sup_{t\geq 0} |T(t)x|_k.$$

§ 4. Extension theorem

The theorems formulated in the preceding paragraph can be applied to the study of the following problem.

Let B be a Banach space. Introduce in B a new topology, weaker than the original one, by means of a countable family of seminorms. Let F be the closure of this new locally convex space; F is a Fréchet space. Let $\{T(t); t \geq 0\}$ be a strongly continuous semigroup of operators in B. The question is as follows: under what conditions can $T(t)$ be extended, preserving the property of strong continuity, to the whole space F? The answer follows directly from Theorems 3 and 4.

Theorem 5. *Let $T(t)$ be a strongly continuous semigroup of operators in the Banach space B; let F be a Fréchet space containing B as a subset, and let B (regarded as a subset of F) be everywhere dense in F.*

Then, in order that $T(t)$ admit an extension to a strongly continuous semigroup in F (not necessarily satisfying condition ()), it is necessary and sufficient that condition (8) hold for $\Sigma = B$ and $X = F$.*

Corollary. In order that $T(t)$ admit an extension from B to F **with condition (*) fulfilled**, it is necessary and sufficient that condition (8) be satisfied for $\vartheta_k(t) = c_k$ (= const).

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Received
27 X 1961

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Note: Figure translations are in progress. See original paper for figures.