ON THE BUCKLING SURFACES OF THIN SHELLS UNDER LOCAL LOSS OF STABILITY

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Abstract

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ON THE BUCKLING SURFACES OF THIN SHELLS UNDER LOCAL LOSS OF STABILITY

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In papers ([6,7]), on the basis of the concept of the isometry of buckling surfaces, a geometric theory of postcritical deformations of thin shells was developed. Below we investigate the buckling shapes of a circular cylindrical shell under axial compression, of a circular conical shell compressed along its generators, and of a closed spherical shell under uniform external pressure. The usual method for solving nonlinear stability problems is the energy method or the Bubnov–Galerkin method. The proposed approach is based on separating the problem of finding the buckling shape from the problem of determining the critical load. On the basis of the theory of local stability developed in papers ([4,5]), this method, in a modified form, was applied by V. Z. Vlasov ([1]).

Under local loss of stability of shells, a large number of waves is formed on their surfaces. Solutions of the indicated problems in the linear formulation, as well as numerous nonlinear solutions, show that the upper and lower critical stresses are proportional to the relative thickness of the shell and do not depend on the character of the boundary conditions, if the buckling zone does not adjoin the fastening contour. Physically this is an obvious fact. The influence of the boundary conditions in this case is damped by numerous longitudinal harmonics, and the critical load turns out not to depend on the length. Approximate solutions on the stability of a cylindrical shell under axial compression in the nonlinear formulation have been carried out precisely under the assumption that the buckling shape does not satisfy any boundary conditions formulated in advance ([8–10]).

To solve the problems posed, we shall adopt as the principal assumption the independence of the critical load from the character of the boundary conditions. In other words, we shall assume that the critical load does not depend on equilibrium forms satisfying the original system of equations, and that to the set of such forms there corresponds a critical load unique in modulus.

We solve the system of nonlinear equations ([2])

$$\nabla^2\nabla^2 F - Et\left(\frac{1}{R_2}W_{xx} + \frac{1}{R_1}W_{yy}\right) + EtL(W) = 0, \qquad L(W) = W_{xx}W_{yy} - W_{xy}^2,$$

$$\nabla^2\nabla^2 W + \frac{1}{D}\left(\frac{1}{R_2}F_{xx} + \frac{1}{R_1}F_{yy}\right) - \frac{1}{D}\left(F_{xx}W_{yy} + F_{yy}W_{xx} - 2F_{xy}W_{xy}\right) + \frac{P}{D} = 0 \tag{1}$$

respectively for the cylinder and the sphere in the form:

$$\text{a)}\ \ F = i\left(AW - T_0\frac{y^2}{2}\right); \qquad \text{b)}\ \ F = i\left[AW - \frac{T_0(x^2+y^2)}{2}\right], \quad i = \sqrt{-1}; \tag{2}$$

A is an arbitrary constant, T_0 is the compressive force. Here the notation customary in shell theory is adopted ([1–3]).

For the sphere we consider the initial state to be momentless. Therefore in (1), for the sphere, one should set $T_0 = -PR/2$. Here the equations have been written for

of thin shells. However, the method is applicable and leads to the same results if the equations take into account the variation of the coefficients of the first and second quadratic forms of the surface. The form of the solutions (2) reflects a physically visual picture of bulging. The functions F, W are surfaces in three-dimensional space. The factor i denotes their phase shift by $\pi/2$, i.e., the maximum of the stresses from F corresponds to the minimum of the change in curvature. On the areas of the dents the changes in curvature are minimal, and the stresses are maximal (close to zero). At the places of maximum bending of the shell the compressive stresses are negative, i.e., minimal ([3]).

Let us carry out the solution for a cylindrical shell. Substituting (2) into (1), we obtain the system

$$\nabla^2\nabla^2 W + \frac{Eti}{AR}W_{xx} - \frac{Eti}{A}L(W) = 0,$$

$$\nabla^2\nabla^2 W + \frac{AiW_{xx}}{DR} + \frac{T_0 i}{D}W_{xx} - \frac{2Ai}{D}L(W) = 0 \tag{3}$$

and its compatibility condition

$$\left(T_0R + A - \frac{EtD}{A}\right)W_{xx} - \left(2A - \frac{EtD}{A}\right)\left(W_{xx}W_{yy} - W_{xy}^2\right) = 0. \tag{4}$$

In view of the arbitrariness of A and T_0, we set the brackets in (4) equal to zero. We obtain

$$A = \pm\sqrt{\frac{EtD}{2}}, \qquad T_0 R = \frac{EtD}{A} - A \tag{5}$$

(the minus sign for A does not satisfy the requirement of positivity of T_0).

Since the boundary conditions are immaterial, (5) determines the critical load

$$T_0 R = \frac{\sqrt{EtD}}{\sqrt{2}} \quad \text{or} \quad \sigma_{\text{cr}} = \frac{0.204Et}{R(1-\nu^2)^{1/2}}. \tag{6}$$

For a spherical shell the procedure outlined leads to the same formula (6). For a truncated conical shell, assuming the initial state to be momentless, we find ([2])

$$T_0 r_0 \tan\gamma = \sqrt{EtD}/\sqrt{2}, \tag{6'}$$

where r_0 is the distance along the generator from the vertex of the cone to the small end section; γ is half the cone angle at the vertex; T_0 is the force referred to the section $r = r_0$.

We shall show that the stresses (6) and (6′) correspond to a real state of the shell, and not an imaginary one, as might be concluded from (2). Transforming (1) by means of the substitutions

$$W = \frac{(1+i)}{R}\frac{y^2}{2} - i\psi,$$

$$F = A\psi - \frac{T_0 y^2}{2}, \tag{7}$$

we obtain the system

$$\nabla^2\nabla^2\psi + \frac{Et}{AR}\psi_{xx} - \frac{Et}{A}L(\psi) = 0,$$

$$\nabla^2\nabla^2\psi + \frac{A\psi_{xx}}{DR} + \frac{T_0\psi_{xx}}{D} - \frac{2A}{D}L(\psi) = 0, \tag{8}$$

whose compatibility conditions coincide with (4), (5), and (6). Equations (8) and the second equality in (7), taking into account (5) and (6), lead to the equation and equal-

to a system with real coefficients

$$\nabla^2\nabla^2\psi + \sqrt{\frac{2Et}{D}}\left(\frac{\psi_{xx}}{R} + \psi_{xy}^2 - \psi_{xx}\psi_{yy}\right) = 0,$$
$$F = \sqrt{\frac{EtD}{2}}\left(\psi - \frac{y^2}{2R}\right). \tag{9}$$

Similarly, equations (3), as well as the first of equalities (2), taking (5) and (6) into account, are reduced to the form

$$\nabla^2\nabla^2 W + i\sqrt{\frac{2Et}{D}}\left(\frac{W_{xx}}{R} + W_{xy}^2 - W_{xx}W_{yy}\right) = 0,$$
$$F = i\sqrt{\frac{EtD}{2}}\left(W - \frac{y^2}{2R}\right). \tag{10}$$

From comparison of (9) and (10) we conclude that F is a real function corresponding to the real load (6). Let us show that there exists a real stressed state due to bending. The first of substitutions (7), in the case under consideration with $1/R_1 = 0$ and $P = 0$, preserves the structure of the second equilibrium equation of system (1) unchanged. Since, by virtue of the assumption adopted, the boundary conditions for the normal deflection are immaterial, the function ψ is an actual form of elastic equilibrium characterizing a real bending state.

The method does not allow us to judge whether the loads (6), (6′) are an absolute minimum. However, they are very close to the lower critical loads determined by approximate solutions of these problems ([2], [3], [8—10]).

The coincidence of the coefficients in formulas (6), (6′) for the cylinder and the cone was previously proved by another method in the paper ([11]). Let us prove that the form of buckling of a cylindrical shell under axial compression at the load (6), close to the lower critical one, is a surface similar to an isometric one. The assertion is also valid for spherical and conical shells.

Let T_0 exceed the critical load (6) by some small but finite quantity. This is possible if A takes the value

$$A_1 = \sqrt{\frac{EtD}{2}}(1-\varepsilon_1), \qquad \varepsilon_1 = \varepsilon\Big/\sqrt{\frac{EtD}{2}} > 0. \tag{11}$$

Then $T_0R + A - EtD/A \simeq -3E$, $2A_1 - EtD/A_1 \simeq -4\varepsilon$, and equation (4) for W (and similarly for ψ) will be

$$\frac{3}{4}\frac{W_{xx}}{R} + W_{xx}W_{yy} - W_{xy}^2 = 0. \tag{12}$$

By means of the substitution $W = {}^3/_4 W_1$ it is reduced to the equation of the isometric surface W_1.

Thus, the load (6) is a bifurcation point of the solutions (9) and (10), since the latter, taking (12) into account, split into two:

$$\nabla^2\nabla^2\psi + \frac{7}{3}\sqrt{\frac{2Et}{D}}\,\frac{\psi_{xx}}{R} = 0, \qquad \frac{3}{4}\frac{\psi_{xx}}{R} + \psi_{xx}\psi_{yy} - \psi_{xy}^2 = 0;$$

$$\nabla^2\nabla^2 W + i\frac{7}{4}\sqrt{\frac{2Et}{D}}\,\frac{W_{xx}}{R} = 0, \qquad \frac{3}{4}\frac{W_{xx}}{R} + W_{xx}W_{yy} - W_{xy}^2 = 0. \qquad (13)$$

In the theory developed in ([6]), by means of an equation of the type of the first linear equation (13), the potential energy of deformation and the work of external forces on displacements satisfying the equation of isometric surfaces are taken into account.

Since closed surfaces are characterized by local loss of stability (at least in one direction), it is natural that, when solving the problem by the energy method, an equation of type (12) should be taken as the quali-

basis for choosing approximating functions, determining from it, in the first approximation, the unknown parameters of the deflection function by means of the Bubnov–Galerkin procedure. Thus, for these parameters the solution of systems of nonlinear algebraic equations is very substantially facilitated.

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Note: Figure translations are in progress. See original paper for figures.