ON THE FOURIER TRANSFORMS OF CERTAIN CLASSES OF GENERALIZED FUNCTIONS

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

N. S. LANDKOF

ON THE FOURIER TRANSFORMS OF CERTAIN CLASSES OF GENERALIZED FUNCTIONS

(Presented by Academician V. I. Smirnov on 9 V 1962)

1. Let M_2 denote the class of measurable functions $F(\omega)$ for which

$$\sup_{-\infty<T<\infty} \frac{1}{T} \int_0^T |F(\omega)|^2 \, d\omega < \infty. \tag{1}$$

From the results of N. Wiener ([1,2]) it follows that the Bochner–Plancherel transform

$$B(x) \equiv \int_{-1}^{1} F(\omega) \frac{e^{-2\pi i \omega x} - 1}{-2\pi i \omega} \, d\omega + \operatorname*{l.\,i.\,m.}_{N\to\infty} \int_{1<|\omega|<N} F(\omega) \frac{e^{-2\pi i \omega x}}{-2\pi i \omega} \, d\omega \tag{2}$$

for any function $F(\omega) \in M_2$ has the following property:

$$\int_{-\infty}^{\infty} |B(x+\varepsilon) - B(x-\varepsilon)|^2 \, dx = O(\varepsilon) \quad (\varepsilon \to 0, \infty). \tag{3}$$

We shall denote by Δ_2 the class of locally summable functions satisfying condition (3), and by Δ_2' the class of generalized functions that are derivatives (in the sense of the theory of generalized functions) of elements of Δ_2. From the preceding it follows that the generalized Fourier transform of a function $F(\omega) \in M_2$ belongs to Δ_2'.

We shall prove that in this way the entire class Δ_2' is obtained and thereby give an effective characterization of the Fourier transforms of the elements of Δ_2'. In addition, it turns out that analogues of two well-known theorems of Wiener and Paley ([3]) hold; these will be indicated in § 3.

It is interesting to note that the classes Δ_2' and M_2 (more precisely, their subclasses $\Delta_2'(0,\infty)$ and M_2^-, defined in § 3) can be naturally introduced into the general theory of linear filters, which substantially simplifies the latter. We shall dwell on this question in another note.

2. Theorem 1. *If $f(x) \in \Delta_2'$, then there exists a function $F(\omega) \in M_2$ whose generalized Fourier transform is $f(x)$.*

Proof. We shall prove the following assertion, equivalent to Theorem 1: if $\Phi(x) \in \Delta_2$, then there exists a function $F(\omega) \in M_2$ for which the Bochner-Plancherel transform (2) is equivalent, up to an additive constant, to the function $\Phi(x)$.

Put

$$\Phi_\eta(x) = \frac{1}{2\eta} \int_{x-\eta}^{x+\eta} \Phi(t)\, dt.$$

Since $\Phi_\eta(x)$ is absolutely continuous and $\Phi_\eta'(x) = \dfrac{1}{2\eta}\,[\Phi(x+\eta) - \Phi(x-\eta)] \in L^2$, it follows that $\Phi_\eta'(x) = \widetilde{F_\eta}(\omega)^*$, where $F_\eta(\omega) \in L^2$, and for $\varepsilon > 0$

$$\Phi_\eta(x+\varepsilon) - \Phi_\eta(x-\varepsilon) = \text{l. i. m.}_{N\to\infty} \int_{-N}^{N} F_\eta(\omega)\, \frac{\sin 2\pi\varepsilon\omega}{\pi\omega}\, e^{-\pi i \omega x}\, d\omega.$$

By virtue of a known theorem,

$$\text{l. i. m.}_{\eta\to 0}[\Phi_\eta(x+\varepsilon) - \Phi_\eta(x-\varepsilon)] = \Phi(x+\varepsilon) - \Phi(x-\varepsilon),$$

and therefore there exists

$$F(\omega, \varepsilon) = \text{l. i. m.}_{\eta\to 0}\, F_\eta(\omega)\, \frac{\sin 2\pi\varepsilon\omega}{\pi\omega}.$$

It is not difficult to show, by choosing in a suitable way a sequence $\eta_k \to 0$, that $F(\omega, \varepsilon)$ has the form

$$F(\omega) \frac{\sin 2\pi\varepsilon\omega}{\pi\omega},$$

where $F(\omega)$ is defined almost everywhere. After this, Parseval' s equality

$$\int_{-\infty}^{\infty} |\Phi(x+\varepsilon) - \Phi(x-\varepsilon)|^2\, dx = \int_{-\infty}^{\infty} |F(\omega)|^2 \frac{\sin^2 2\pi\varepsilon\omega}{\pi^2\omega^2}\, d\omega$$

makes it possible to prove that $F(\omega) \in M_2$, and the formula

$$\Phi(x+\varepsilon) - \Phi(x-\varepsilon) = \text{l. i. m.}_{N\to\infty} \int_{-N}^{N} F(\omega)\, \frac{\sin 2\pi\varepsilon\omega}{\pi\omega} e^{-2\pi i \omega x}\, d\omega$$

makes it possible to establish that almost everywhere

$$\Phi(x) = B(x) + \text{const.}$$

The theorem proved admits a generalization. Put

$$T^{\alpha\beta} = \begin{cases} |T|^{\alpha}, & \text{for } 0 \le |T| \le 1, \\ |T|^{\beta}, & \text{for } 1 \le |T| < \infty, \end{cases} \qquad (\alpha, \beta \ge 0)$$

and denote by $M_2^{(\alpha,\beta)}$ the class of measurable functions $F(\omega)$ for which **

$$\int_0^T |F(\omega)|^2 \, d\omega = O(T^{\alpha,\beta}) \,. \tag{4}$$

Since

$$\int_0^T |F(\omega)| \, d\omega = O\Big(T^{\frac{\alpha+1}{2}, \frac{\beta+1}{2}}\Big) \,,$$

then, taking a natural number $n \ge \dfrac{\beta+1}{2}$, we can form the n-th pre-

* $\widetilde{F_\eta}(\omega)$ denotes the Fourier transform of the function $F_\eta(\omega)$.

** Here and in subsequent formulas the symbol $O(\varphi(x))$ is defined by the inequality
$O(\varphi(x)) < \text{const} \cdot \varphi(x)$ $(\varphi(x) \ge 0)$, which must hold throughout the whole domain of definition of $\varphi(x)$.

the Bochner–Plancherel transform:

$$B_n(x) = \int_{-1}^{1} F(\omega) \, \frac{e^{-2\pi i\omega x} - Q_{n-1}(-2\pi i\omega x)}{(-2\pi i\omega)^n} \, d\omega +$$

$$+ \operatorname*{l.\,i.\,m.}_{N\to\infty} \int_{1 \le |\omega| < N} F(\omega) \, \frac{e^{-2\pi i\omega x}}{(-2\pi i\omega)^n} \, d\omega,$$

where

$$Q_{n-1}(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}.$$

Denote by $\delta_\varepsilon^{(n)} B_n(x)$ the n-th central difference of the function $B_n(x)$ with step ε. Then it can be shown that $\delta_\varepsilon^{(n)} B_n(x) \in L^2$ and $\|\delta_\varepsilon^{(n)} B_n(x)\|^2 = O(\varepsilon^{2n-\beta,\, 2n-\alpha})$ under the condition that $\alpha < 2n$. If, however, $\alpha = 2n$ or $\alpha > 2n$, then as $\varepsilon \to \infty$ we shall have, respectively, $\|\delta_\varepsilon^{(n)} B_n(x)\|^2 = O(\ln \varepsilon)$ or $O(1)$.

Denote by $\Delta_{2,n}^{(\alpha,\beta)}$ $(0 \le \alpha < 2n;\ 0 \le \beta \le 2n-1)$ the class of locally summable functions $B(x)$ for which

$$\delta_\varepsilon^{(n)} B(x) \in L^2, \qquad \|\delta_\varepsilon^{(n)} B(x)\|^2 = O(\varepsilon^{2n-\beta,\, 2n-\alpha}).$$

Then the generalization of Theorem 1 can be formulated as follows:

Theorem 2. If $B(x) \in \Delta_{2,n}^{(\alpha,\beta)}$ $(0 \le \alpha < 2n;\ 0 \le \beta \le 2n-1)$, then the generalized function $B^{(n)}(x)$ is the Fourier transform of some function $F(\omega) \in M_2^{(\alpha,\beta)}$.

3. Denote by $\Delta_2'(0,\infty)$ the totality of all generalized functions in Δ_2' which are equal to zero on the half-axis $(-\infty, 0)$, and by M_2^- the class of functions $F(w)$, $w = \omega + it$, regular in the lower half-plane $t < 0$ and belonging there uniformly to M_2, i.e., such that

$$\sup_{T;\, t \le 0} \frac{1}{T} \int_0^T |F(\omega + it)|^2 \, d\omega < \infty.$$

Using estimates of the Poisson integral for the half-plane, one can prove the following theorems, analogous to the well-known theorems of Wiener and Paley (see ([3]), Theorems V and XII).

Theorem 3. The Fourier transform of a generalized function in $\Delta_2'(0,\infty)$ belongs to M_2^-, and the totality of these Fourier transforms fills the whole class M_2^-.

Theorem 4. If $F(w) \in M_2^-$, then

$$\int_{-\infty}^{\infty} \frac{|\ln |F(\omega)||}{1 + \omega^2} \, d\omega < \infty.$$

Conversely, if on the real axis ω a function $G(\omega) \ge 0$, $G(\omega) \in M_2$, is given, for which

$$\int_{-\infty}^{\infty} \frac{|\ln G(\omega)|}{1 + \omega^2} \, d\omega < \infty,$$

then there exists $F(w) \in M_2^-$, having no zeros inside the lower half-plane, such that $|F(\omega)| = G(\omega)$ almost everywhere.

Among the possible generalizations of Theorem 3, let us note the following: If $f(x) \in \Lambda_{2,n+1}^{(2n+1,2n+1)}$ (or $f(x) \in \Lambda_{2,1}^{(1,1)} \equiv \Lambda_2$) and $f(x) \equiv 0$ on the half-axis $(-\infty, 0)$, then the Fourier transform of the generalized function $f^{(n+1)}(x)$ is a function $F(\omega)$, regular in the lower half-plane and such that:

1) $F(\omega) \in M_2^{(2n+1,2n+1)}$; 2) $\dfrac{F(\omega)}{\omega^n} \in M_2$. Conditions 1) and 2) completely characterize the class of Fourier transforms under consideration.

The author expresses his gratitude to V. A. Marchenko for a valuable discussion of the results of this work.

Received
26 V 1962

CITED LITERATURE

[1] N. Wiener, Acta Math., **55**, 117 (1930). [2] N. Wiener, *The Fourier Integral and Certain of its Applications*, Cambridge, 1933. [3] N. Wiener, R. E. A. C. Paley, *Fourier Transforms in the Complex Domain*, N. Y., 1934.

Note: Figure translations are in progress. See original paper for figures.