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**Abstract**

**Full Text**

**MATHEMATICS**

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## **ON THE QUESTION OF A. I. MARKUSHEVICH' S DUALITY PRINCIPLE**

*(Presented by Academician V. I. Smirnov on 30 X 1961)*

Studying spaces of functions analytic in the open disk, A. I. Markushevich established <sup>(1)</sup> a connection between a complete system of functions  $\{f_n(z) = \lambda_n[F(z, \xi)]\}$  and the uniqueness property of a system of linear continuous functionals  $\{\lambda_n\}$ , known in the literature under the name of the duality principle. For systems of functions of several complex variables this principle was generalized by S. A. Eremin <sup>(2,3)</sup>. From the functional point of view, the nature of the duality principle for spaces in which a basis exists was revealed by M. G. Khaplanov <sup>(4)</sup>, who, considering coordinate spaces of an infinite number of dimensions, established a matrix criterion for completeness of a system of elements, which for spaces of functions analytic in the open disk coincides with A. I. Markushevich' s duality principle.

In the present note the duality principle is established for linear topological spaces. This made it possible to show that the use of certain general considerations of functional analysis makes it possible in a simple way to clarify the foundations of the duality principle, and also to extend it to arbitrary analytic spaces. The latter do not limit the domain of applicability of the duality principle.

Let us introduce some definitions and notation. Linear topological spaces  $X, X', Y$  with the corresponding topologies  $\sigma, \sigma', \tau$  will henceforth be denoted respectively by  $[X, \sigma], [X', \sigma'], [Y, \tau]$ . We shall say that the linear topological spaces  $[X, \sigma]$  and  $[X', \sigma']$  form a dual system if the set of all linear continuous functionals in  $[X, \sigma]$  coincides with the set  $X'$ , and conversely, the set of all linear continuous functionals in  $[X', \sigma']$  coincides with the set  $X$ , with  $x'(x) = x(x')$ , i.e. the value of the functional  $x' \in [X', \sigma']$  on the element  $x \in [X, \sigma]$  coincides with the value of the functional  $x \in [X, \sigma]$  on the element  $x' \in [X', \sigma']$  (see <sup>(5,6)</sup>).

Let  $[X, \sigma]$  and  $[X', \sigma']$  be linear topological spaces forming a dual system. Further consider a linear topological space  $[Y, \tau]$ , in which the Hahn–Banach theorem on the extension of linear continuous functionals holds. We shall denote the set of all linear continuous functionals in the space  $[Y, \tau]$  by  $Y'$ .

Let  $A$  be an arbitrary linear continuous operator mapping the space  $[X', \sigma']$  into the space  $[Y, \tau]$ , and let  $A^*$  be the adjoint operator. We shall denote the

ranges of the operators  $A$  and  $A^*$  respectively by  $R_A$  and  $R_{A^*}$ .

A system of elements  $\{y_n = Ax'_n\}$  belonging to  $R_A$  ( $x'_n \in [X', \sigma']$ ),  $n = 0, 1, \dots$ , will be called relatively complete in  $R_A$  if  $R_A$  belongs to the closed linear hull of the system  $\{y_n\}$ .

We shall say that the system of elements  $\{x'_n\}$  of the space  $[X', \sigma']$  has the property of relative uniqueness on  $R_{A^*}$ , if from the fact that

$x \in R_{A^*}$  and  $x'_n(x) = 0$ ,  $n = 0, 1, \dots$ , it follows that  $x = 0$ .

**Theorem 1.** *The system of elements  $\{y_n = Ax'_n\}$  ( $x'_n \in [X', \sigma']$ ) is relatively complete in  $R_A$  if and only if the system of elements  $\{x'_n\}$  has the property of relative uniqueness on  $R_{A^*}$ .*

**Proof.** It is obvious that the system of elements  $\{y_n = Ax'_n\}$  ( $x'_n \in [X', \sigma']$ ) will be relatively complete in  $R_A \subset [Y, \tau]$  if and only if from the fact that  $y' \in Y'$  and  $y'(Ax'_n) = 0$ ,  $n = 0, 1, \dots$ , it follows that  $y'(Ax') = 0$  for all  $x' \in [X', \sigma']$ . This means that the system of elements  $\{Ax'_n\}$  will be relatively complete in  $R_A$  if and only if from the fact that  $y' \in Y'$  and

$$y'(Ax'_n) = A^*y'(x'_n) = x'_n(A^*y') = 0, \quad n = 0, 1, \dots,$$

it follows that

$$y'(Ax') = A^*y'(x') = x'(A^*y') = 0$$

for all  $x' \in [X', \sigma']$ , i.e.  $A^*y' = 0$ . Since  $A^*y'$  is an arbitrary element of  $R_{A^*}$ , the theorem is proved.

**Corollary.** If  $R_A = Y$ ,  $R_{A^*} = X$ , then the system of elements  $\{Ax'_n\}$  is complete in  $[Y, \tau]$  if and only if the system of elements  $\{x'_n\}$  has the property of uniqueness on  $[X, \sigma]$ .

We shall now formulate the duality principle of A. I. Markushevich for spaces of functions analytic in an arbitrary (closed or open) set of the Riemann sphere  $\Omega$ .

For definitions and notation see the paper <sup>(7)</sup>. We specify only that by  $[\mathfrak{S}_n(\mathfrak{D})]$  (respectively, by  $[\mathfrak{H}_n(\mathfrak{A})]$ ) we denote the sequence of sets used in defining the topology in the space  $P(\mathfrak{D})$  ( $R(\mathfrak{A})$ ) (<sup>(7)</sup>; pp. 32, 38). An analogous meaning is also attached to the notation  $[\mathfrak{H}_n(\Omega - \mathfrak{D})]$ ,  $[\mathfrak{S}_n(\Omega - \mathfrak{A})]$ , etc.

Consider two arbitrary analytic spaces  $P(\mathfrak{D}_1)$  and  $P(\mathfrak{D}_2)$  of the same type as the space  $P(\mathfrak{D})$ , and formulate for them the duality principle. Let  $a(z_1, z_2)$  be a function of two complex variables, locally analytic on  $\mathfrak{S}_n(\mathfrak{D}_1) \times \mathfrak{S}_m(\mathfrak{D}_2)$  for all natural  $n$  and  $m$ . Consider the expression

$$Au_1 = u_1[a(z_1, z_2)] = \int_{C_{u_1}} u_1(t_1)a(t_1, z_2) dt_1,$$

where  $u_1 \in R(\Omega - \mathfrak{D}_1)$ , and, consequently,  $u_1$  is a locally analytic function on some set  $\mathfrak{H}_n(\Omega - \mathfrak{D}_1)$ ; as the path of integration  $C_{u_1}$  we choose the boundary of the set  $\mathfrak{H}_{n+1}(\Omega - \mathfrak{D}_1)$ .

According to (7,8),  $A$  is a linear continuous operator mapping the space  $R(\Omega - \mathfrak{D}_1)$  into the space  $P(\mathfrak{D}_2)$ . The operator conjugate to the operator  $A$  has the form

$$A^*u_2 = u_2[a(z_1, z_2)] = \int_{C_{u_2}} u_2(t_2)a(z_1, t_2) dt_2,$$

where  $u_2 \in R(\Omega - \mathfrak{D}_2)$ , and the path of integration  $C_{u_2}$  is chosen in an analogous manner.

We note that the spaces  $P(\mathfrak{D}_1)$  and  $R(\Omega - \mathfrak{D}_1)$ ,  $P(\mathfrak{D}_2)$  and  $R(\Omega - \mathfrak{D}_2)$  are in duality ((7), p. 40). Thus all the assumptions of the preceding theorem are fulfilled.

**Theorem 2.** *The system of functions  $\{u_1^{(n)}[a(z_1, z_2)]\}$  is relatively complete in*

$$R_A = \{u_1[a(z_1, z_2)]\}_{u_1 \in R(\Omega - \mathfrak{D}_1)},$$

*if and only if the system of functionals  $\{u_1^{(n)}\}$  has the property of relative uniqueness on*

$$R_{A^*} = \{u_2[a(z_1, z_2)]\}_{u_2 \in R(\Omega - \mathfrak{D}_2)}.$$

This assertion also holds for pairs of spaces:

- 1)  $P(\mathfrak{D}_1)$  and  $R(\mathfrak{A}_2)$ ;    2)  $R(\mathfrak{A}_1)$  and  $P(\mathfrak{D}_2)$ ;    3)  $R(\mathfrak{A}_1)$  and  $R(\mathfrak{A}_2)$ .

The only difference is that different conditions (8) are imposed on the function of two variables in each of the cases under consideration. Thus, in the first case one considers a function  $a(z_1, \lambda_2)$ , locally analytic on  $\mathfrak{G}_n(\mathfrak{D}_1) \times \mathfrak{H}_{m(n)}(\mathfrak{A}_2)$  for  $n = 1, 2, \dots$  and  $m$  depending on  $n$ . In the second case, the condition of local analyticity on  $\mathfrak{H}_{n(m)}(\mathfrak{A}_1) \times \mathfrak{G}_m(\mathfrak{D}_2)$  is imposed on the function  $a(\lambda_1, z_2)$  for  $m = 1, 2, \dots$  and  $n$  depending on  $m$ . Finally, in the third case the function of two complex variables  $a(\lambda_1, \lambda_2)$  must be locally analytic on at least one of the sets of the form  $\mathfrak{H}_n(\mathfrak{A}_1) \times \mathfrak{H}_m(\mathfrak{A}_2)$ .

As an example of nonanalytic spaces for which the duality principle holds, one may consider the following two spaces:  $H$  is the space of real functions  $f(x)$  differentiable on the interval  $[0, 1]$  such that

$$\|f\|^2 = \int_0^1 [f'(x)]^2 dx < \infty, \quad f(0) = 0;$$

$L_2$  is the space of real square-summable functions  $g(t)$  on the interval  $[0, 1]$ , and

$$\|g\|^2 = \int_0^1 [g(t)]^2 dt < \infty.$$

Let  $a(x, t)$  be a function of two variables possessing the following properties: for every  $t \in [0, 1]$ ,  $a(x, t) \in H$  and

$$\int_0^1 \left[ \frac{\partial}{\partial x} a(x, t) \right]^2 dx \leq [\rho(t)]^2,$$

where  $\rho(t)$  is some function from  $L_2$ ; for every  $x \in [0, 1]$ ,  $a(x, t) \in L_2$  and  $\frac{\partial}{\partial x} a(x, t) \in L_2$ . Then the operator

$$Af = \int_0^1 f'(x) \frac{\partial}{\partial x} a(x, t) dx \quad (f(x) \in H)$$

is a linear continuous operator mapping the space of functionals  $H' = H$  into the space  $L_2$ .

The operator adjoint to the operator  $A$  acts from the space  $L'_2$  into the space  $H$  and has the form

$$A^*g = \int_0^x \int_0^1 \frac{\partial}{\partial x} [a(x, t)] \Big|_{x=\tau} g(t) dt d\tau \quad (g(t) \in L'_2).$$

The duality principle is formulated for these spaces in an obvious way.

In conclusion, I express my deep gratitude to K. M. Fishman for suggesting the topic and for the help rendered to me in the course of the work.

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*Note: Figure translations are in progress. See original paper for figures.*

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