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Abstract

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ON SOME PROBLEMS OF HARMONIC ANALYSIS

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In the present note we continue the investigation of some questions of harmonic analysis begun in the author' s paper ([1]). Let us briefly recall the statement of the problem and the basic definitions.*

In the space $\mathcal{H}$ any set $M \subset R_\nu$ with $\mu(M) > 0$ makes it possible to define the special norm

$$\|\hat{u}(\xi)\|_M = \left(\int_M |\hat{u}(\xi)|^2 \, d\xi\right)^{1/2}, \tag{1}$$

weaker than the usual norm $\|\cdot\|$. The problem is to find such sets M that generate a norm $\|\cdot\|_M$ equivalent to the norm $\|\cdot\|$. If we denote

$$\gamma(\mathfrak{M}) = \sup_{\hat{u} \in E} \left(\int_{\mathfrak{M}} |\hat{u}(\xi)|^2 \, d\xi\right)^{1/2}, \tag{2}$$

then this problem is equivalent to describing those sets $\mathfrak{M}$ (called determining in ([1])) for which $\gamma(\mathfrak{M}) < 1$. In ([1]), along with a series of sufficient criteria for determining sets, a condition was also formulated that is necessary in order that $\gamma(\mathfrak{M}) < 1$ hold (Theorem 6). Below it will be shown that in the space R_1 this condition is also sufficient, which completely solves the problem posed in the one-dimensional case. If the number of variables $\nu \geqslant 2$, then we give a sufficient criterion for a determining set, more general than those given in ([1,2]). At the end of the article a curious variational problem is formulated, and its solution is given in a number of particular cases.

1. In ([1]) the notion of the asymptotic density of a set was introduced:

$$\beta(\mathfrak{M}) = \lim_{n\to\infty} \left[\sup_{\xi \in R_\nu} \frac{\mu(\mathfrak{M} \cap K_n^\xi)}{n^\nu}\right] \tag{3}$$

(here K_n^ξ is the cube with center at the point $\xi \in R_\nu$ and edges of length n, parallel to the coordinate axes).** We shall now consider a new characteristic of a set, more delicate than asymptotic density.

Definition 1. To each set $\mathfrak{M} \subset R_\nu$ we assign the nonincreasing function

$$S_{\mathfrak{M}}(r) = \inf\left\{ n : \sup_{\xi\in R_\nu} \mu(\mathfrak{M} \cap K_n^\xi) \leqslant n^\nu - r \right\}.$$

* We retain the basic notation of ([1]). In particular, $\mathcal{H}$ denotes the Hilbert space of entire functions $\hat{u}(\xi_1, \dots, \xi_\nu) = Fu(x^1, \dots, x^\nu)$ with norm

$$\|\hat{u}(\xi)\| = \left(\int_{R_\nu} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2},$$

where F is the Fourier operator, and $u(x^1, \dots, x^\nu)$ is a finite function, square-summable in the bounded domain Ω. E is the unit sphere of the space $\mathcal{H}$. The sets $\mathfrak{M}$ and M are always regarded as complementary to each other in R^ν, i.e. $\mathfrak{M} \cup M = R_\nu$. $\mu(\mathcal{A})$ is the Lebesgue measure of the set $\mathcal{A}$.

** It is easy to show that this limit always exists.

and denote

$$S(\mathfrak{M}) = \lim_{r\to+0} S_{\mathfrak{M}}(r). \tag{4}$$

The following lemma establishes a simple relation between the numbers $S(\mathfrak{M})$ and $\beta(\mathfrak{M})$.

Lemma 1. In order that the inequality $\beta(\mathfrak{M}) < 1$ hold, it is necessary and sufficient that $S(\mathfrak{M}) < \infty$.

Proof is almost obvious. Denote $\mu(\mathfrak{M} \cap K_n^\xi) = \psi(n, \xi)$. If $S(\mathfrak{M}) < \infty$, then for some $n_0 > S(\mathfrak{M})$

$$\sup_{\xi\in R_\nu} \psi(n_0, \xi) = n_0^\nu - r, \quad r > 0.$$

It is clear that, for integral m,

$$\sup_{\xi\in R_\nu} \frac{\psi(mn_0, \xi)}{m^\nu n_0^\nu} = 1 - \frac{r}{n_0^\nu},$$

and, letting here $m \to \infty$, we find, according to (3), that $\beta(\mathfrak{M}) < 1$.

Conversely, let $\beta(\mathfrak{M}) < 1$. Then, for all $n \geq n_0$,

$$\sup_{\xi} \frac{\psi(n, \xi)}{n^\nu} \leq 1 - r, \quad r > 0,$$

whence

$$\sup_{\xi} \psi(n, \xi) \leq n^{\nu} - rn^{\nu}.$$

According to the definition, this also means that $S(\mathfrak{M}) \leq n_0 < \infty$.

Let us formulate one more lemma, easily proved by contradiction.

Lemma 2. If an expanding sequence of sets

$$\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \ldots \subset \mathfrak{M}$$

exhausts the set $\mathfrak{M}$ completely, then

$$\lim_{n\to\infty} \gamma(\mathfrak{M}_n) = \gamma(\mathfrak{M}).$$

Theorem 1. In order that the set $\mathfrak{M} \subset R_\nu$ be a determining set, it is necessary, and in the case $\nu = 1$ also sufficient, that the asymptotic density $\beta(\mathfrak{M}) < 1$.

Proof. Necessity. Suppose that $\beta(\mathfrak{M}) = 1$. Then, by Lemma 1, $S(\mathfrak{M}) = \infty$, and consequently there exists a sequence of cubes $K_n^{\xi_n}$ for which

$$\mu(M \cap K_n^{\xi_n}) \to 0, \quad \text{as } n \to \infty.$$

According to Lemma 1 ([1]), $\gamma(M \cap K_n^{\xi_n}) \to 0$, and since, by Lemma 2 of the present paper and (1),

$$\gamma(K_n^{\xi_n}) \xrightarrow[n\to\infty]{} 1,$$

we find that

$$\gamma(\mathfrak{M}) = \lim \gamma(\mathfrak{M} \cap K_n^{\xi_n}) \geq \lim \gamma(K_n^{\xi_n}) - \lim \gamma(M \cap K_n^{\xi_n}) = 1.$$

This proves the first part of the theorem.

Sufficiency. Without loss of generality one may assume that the fundamental domain Ω is the interval $[-\pi, \pi]$. Let $\mathfrak{M}$ be a set on the line

$$R_1 = \{\xi : |\xi| < \infty\}$$

and let $\beta(\mathfrak{M}) < 1$. Then $S(\mathfrak{M}) < \infty$, and hence, for some $N > S(\mathfrak{M})$, any interval of length N intersects $\mathfrak{M}$ in a set of measure less than $N - d$, where $d = \text{const} > 0$. Divide the line R_1 into intervals e_n of length N so that

$$e_n = [nN, (n+1)N], \quad n = 0, \pm 1, \ldots.$$

Each interval e_n, in turn, we divide into

$$K = \left[\frac{2N^2\tau}{d}\right] + 1$$

equal intervals

$$e_{n,p}, \quad p = 1, \ldots, K.$$

A simple count shows that among the intervals $e_{n,p}$ (with n fixed) there will be at least $2N\tau$ such intervals* for which

$$\mu(M \cap e_{n,p}) > \frac{1}{KT},$$

where

$$T = K - 2N\tau + 1.$$

In the collection composed of the latter intervals, choose every second interval and enumerate the remaining ones from left to right:

$$e_n^{(1)}, \dots, e_n^{(N\tau)}.$$

Carry out a similar procedure with each interval e_n, $n = 0, \pm 1, \dots$. Now consider the sequence

$$\{\xi_k\}_{-\infty}^{\infty} = \{\xi_{nN\tau+s}\}_{n=-\infty,\ s=1,\dots,N\tau}^{\infty},$$

where $\xi_{nN\tau+s}$ is an arbitrary point of the set $\mathfrak{M} \cap e_n^{(s)}$. It follows directly from the construction that

$$|\xi_k - \xi_{k'}| \geq \frac{1}{KT} \quad \text{for any } k \neq k'. \tag{5}$$

* One may assume, of course, that the number $N\tau$ is an integer.

Moreover,

$$\left|\xi_k - \frac{k}{\tau}\right| = \left|\xi_{nN\tau+s} - \frac{nN\tau + s}{\tau}\right| \leqslant |\xi_{nN\tau+s} - nN| \leqslant N, \tag{6}$$

since, for any $s = 1, \dots, N\tau$, the point $\xi_{nN\tau+s} \in e_n$. Let now $\hat{u}(\xi) \in \mathcal{H}$ be an arbitrary function. Then, in accordance with the choice of the intervals $e_n^{(p)}$, we have

$$\int_M |\hat{u}(\xi)|^2 \, d\xi = \sum_n \int_{M \cap e_n} |\hat{u}(\xi)|^2 \, d\xi \geqslant \sum_{p=1}^{N\tau} \sum_{n=-\infty}^{\infty} \int_{M \cap e_n^{(p)}} |\hat{u}(\xi)|^2 \, d\xi \geqslant$$

$$\geqslant \frac{1}{KT} \sum \sum_{\xi \in M \cap e_n^{(p)}} \inf |\hat{u}(\xi)|^2 = \frac{1}{KT} \sum_{p=1}^{N\tau} \sum_{n=-\infty}^{\infty} |\hat{u}(\lambda_{n,p})|^2, \tag{7}$$

where $\lambda_{n,p} \in M \cap e_n^{(p)}$. Since the sequence $\{\lambda_{n,p}\}$, as was proved, satisfies conditions (5)–(6), the theorem of Duffin and Schaeffer ([3]) on the equivalence in $\mathcal{H}$ of the norm generated by the last sum in (7) and the norm $\|\cdot\|$ is applicable. This gives

$$\int_M |\hat{u}(\xi)|^2\,d\xi \geqslant \frac{A}{KT}\int_{-\infty}^{\infty}|\hat{u}(\xi)|^2\,d\xi, \qquad A = \text{const} > 0, \tag{8}$$

where A here does not depend on $\hat{u}(\xi) \in \mathcal{H}$. From (8) it follows immediately that

$$\|\hat{u}(\xi)\|^2_{\mathfrak{M}} \leqslant \left(1 - \frac{A}{KT}\right)\|\hat{u}(\xi)\|^2,$$

whence we finally find that

$$\gamma(\mathfrak{M}) \leqslant \left(1 - \frac{A}{KT}\right)^{1/2}.$$

The theorem is proved.

2. In this item we shall describe a class of determining sets in the space $R_{\nu>2}$, including all sets known up to now ([1,2]) with $\gamma(\mathfrak{M}) < 1$.

Definition 2. Let a family of functions $\{f_\alpha(x)\}$, $\alpha \in A$, be given on a set T. We shall say that a point $x_0 \in T$ is a **growth point** of the family $\{f_\alpha\}$ if the set of numbers $\{f_\alpha(x_0)\}$ is unbounded. The lower boundary of the set of all growth points of the family $\{f_\alpha(x)\}$ will be denoted by $\sigma\{f_\alpha\}$.

Denote by $\mathfrak{M}(\xi')$ the section of the set $\mathfrak{M}$ by the line $\xi' = (\xi_2, \dots, \xi_\nu) = \text{const}$.

Theorem 2. Let $\mathfrak{M} \subset R_\nu$. If in the hyperplane $\xi_1 = 0$ there exists a determining (in the space of $\nu - 1$ dimensions) set V such that for $\xi' \in CV$ one has

$$\sigma\{S_{\mathfrak{M}(\xi')}(r)\} > 0,$$

then $\gamma(\mathfrak{M}) < 1^*$.

Remark. It is sufficient, of course, to require that the condition of the theorem be fulfilled for some hyperplane $(\xi, a) = \text{const}$.

The proof of the assertion formulated is based on two lemmas.

Lemma 3. In order that $\gamma(\mathfrak{M}) < 1$, it is necessary and sufficient that, with some constant $c > 0$, the inequality **

$$\|\hat{u}(\xi)\|_{\mathfrak{M}} \leqslant c\,\|\hat{u}(\xi)\|_M, \qquad \hat{u}(\xi) \in \mathcal{H}.$$

be satisfied.

Lemma 4. Let $\{\mathfrak{M}_\alpha\}$ be a collection of sets on the line. In order that the set of numbers $\{\gamma(\mathfrak{M}_\alpha)\}$ be separated from 1, i.e. $\gamma(\mathfrak{M}_\alpha) \leqslant \lambda < 1$, it is necessary and sufficient that

$$\sigma\{S_{\mathfrak{M}_\alpha}(r)\} > 0.$$

Proof of Theorem 2. Denote by $\mathfrak{M}_V$ that part of the set $\mathfrak{M}$ whose orthogonal projection onto the hyperplane $\xi_1 = 0$ coincides with V. By assumption,

$$\sigma\{S_{\mathfrak{M}(\xi')}(r)\} > 0, \qquad \xi' \in CV,$$

so that

* CV denotes the complement of V in the hyperplane $\xi_1 = 0$.

** We recall that $\mathfrak{M} \cup M = R_\nu$.

by Lemma 4, $\gamma(\mathfrak{M}CV(\xi')) \leqslant \lambda < 1$, and, applying Lemma 3, we find that

$$\int_{\mathfrak{M}CV} |\hat{u}(\xi)|^2 \, d\xi \leqslant \lambda \int_{\mathfrak{M}CV} |\hat{u}(\xi)|^2 \, d\xi, \qquad \hat{u}(\xi) \in \mathcal{H}. \tag{9}$$

Next, since, by hypothesis, the set V—which is determining in $R_{\nu-1}$—satisfies Theorem 2 of [1], this, together with Lemma 3, gives

$$\int_{\mathfrak{M}V} |\hat{u}(\xi)|^2 \, d\xi \leqslant \gamma \int_{-\infty}^{\infty} d\xi_1 \int_{CV} |\hat{u}(\xi)|^2 \, d\xi' =$$

$$= \gamma \int_{\mathfrak{M}CV} |\hat{u}(\xi)|^2 \, d\xi + \gamma \int_{\mathfrak{M}CV} |\hat{u}(\xi)|^2 \, d\xi. \tag{10}$$

Comparing (9) and (10), we find that

$$\int_{\mathfrak{M}} |\hat{u}(\xi)|^2 \, d\xi \leqslant (\lambda + \gamma + \lambda\gamma) \int_{\mathfrak{M}} |\hat{u}(\xi)|^2 \, d\xi$$

and the application of Lemma 3 completes the proof of the theorem.

3. The preceding results admit an interesting development connected with the variational problem formulated below, whose solution in the general case will, it seems to us, require the use of methods substantially subtler than those used above.

We shall call a set $\mathfrak{M} \subset R_\nu$ a set of attainability if there exists a function $\hat{u}_0(\xi) \in E$ for which $\|\hat{u}_0(\xi)\|_{\mathfrak{M}} = \gamma(\mathfrak{M})$; in other words, if in the space $\mathcal{H}$ the variational problem

$$\int_{\mathfrak{M}} |\hat{u}(\xi)|^2 \, d\xi = \text{maximum} \qquad \text{under the condition} \qquad \int_{R_\nu} |\hat{u}(\xi)|^2 \, d\xi = 1.$$

is solvable.

It is obvious that, with the exception of the whole space, every set of attainability $\mathfrak{M}$ is determining. The converse is not true, as is easily seen from the example of a half-plane in R_2. The problem is to describe all sets of attainability. The following theorem describes some of them.

Theorem 3. *If $\mu(\mathfrak{M}) < \infty$, then $\mathfrak{M}$ is a set of attainability. If $\mathfrak{M} = \mathfrak{M}_p \times \mathfrak{M}_{\nu-p}$, where $\mathfrak{M}_p \subset R_p$, and each of the sets $\mathfrak{M}_p$ and $\mathfrak{M}_{\nu-p}$ is a set of attainability (in the corresponding space), then $\mathfrak{M}$ is a set of attainability.*

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Note: Figure translations are in progress. See original paper for figures.