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Abstract

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THE DIRICHLET PROBLEM FOR AN EQUATION WITH A SMALL PARAMETER AND DISCONTINUOUS COEFFICIENTS

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Let a domain D with smooth boundary Γ be given in the n-dimensional Euclidean space R^n. Denote by L^ε the following elliptic differential operator, defined and nondegenerate in the domain $D \cup \Gamma \subset R^n$:

$$L^\varepsilon = \varepsilon^2 \left(\frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x^i} \right) + \sum_{i=1}^{n} \tilde{b}_i(x) \frac{\partial}{\partial x^i}. \tag{1}$$

We assume that the coefficients of the operator L^ε belong to the class $C^{(3)}$ everywhere in the domain D, except for an $(n-1)$-dimensional manifold $S \subset D$. On the manifold S, which is assumed to belong to the class $C^{(3)}$, the coefficients of the operator may have a discontinuity of the first kind. Let the function $\psi(x)$ be defined and continuous for $x \in \Gamma$. Consider the following boundary-value problem:

$$L^\varepsilon u^\varepsilon(x) = 0 \qquad \text{for } x \in D \setminus S,$$

$$\lim_{x \to x_0} u^\varepsilon(x) = \psi(x_0) \qquad \text{for } x_0 \in \Gamma, \tag{2}$$

$$u^\varepsilon(x) \text{ and } \operatorname{grad} u^\varepsilon(x) \text{ are continuous for } x \in D.$$

In the present note we formulate some theorems on the behavior of $u^\varepsilon(x)$ as $\varepsilon \to 0$.

Denote by $\sigma(x) = \{\sigma_{ij}(x)\}$ a matrix such that $\{a_{ij}(x)\} = \{\sigma_{ij}(x)\}\{\sigma_{ij}(x)\}^*$ (the asterisk denotes transposition). Outside the domain D, define the functions $\sigma_{ij}(x)$ and $b_i(x)$ arbitrarily, but so that they satisfy the Lipschitz condition everywhere outside S. Consider the stochastic equation

$$x_t^\varepsilon - x_0 = \varepsilon \int_0^t \sigma(x_u^\varepsilon)\, d\xi_u + \varepsilon^2 \int_0^t b(x_u^\varepsilon)\, du + \int_0^t \tilde{b}(x_u^\varepsilon)\, du. \tag{3}$$

Here $\xi_u = \{\xi_u^1, \dots, \xi_u^n\}$ is an n-dimensional Wiener process; $b(x) = \{b_1(x), b_2(x), \dots, b_n(x)\}$; $\tilde{b}(x) = \{\tilde{b}_1(x), \dots, \tilde{b}_n(x)\}$.

As shown in ([1]), equation (3) has a solution. From this solution one constructs a certain Markov process $\widetilde{X}^\varepsilon = \{\tilde{x}_t^\varepsilon, \widetilde{P}_x^\varepsilon\}$ in the space R^n. Denote by $X^\varepsilon = \{x_t^\varepsilon, P_x^\varepsilon\}$ the process obtained from $\widetilde{X}^\varepsilon$ by stopping at the moment of the first exit to the boundary of the domain D. The process X^ε will be a Feller process with continuous trajectories (for the definition of a Markov Feller process see ([2])). For any random variable $\xi(\omega)$, put

$$M_x^\varepsilon \xi(\omega) = \int_\Omega \xi(\omega)\, P_x^\varepsilon(d\omega).$$

Let

$$\tau^\varepsilon(\omega) = \inf\{t : \ x_t^\varepsilon \in \Gamma\}.$$

Theorem 1. The function

$$u^\varepsilon(x) = M_x^\varepsilon \psi\left(x_{\tau_\varepsilon}^\varepsilon\right) \tag{4}$$

is the unique solution of problem (2).

Indeed, as shown in ([3]), problem (2) has a solution in the class of functions having bounded discontinuous second derivatives for $x \in D \setminus S$. With the help of K. Itô' s formula (see ([4,5])) it is proved that such a solution belongs to the domain of definition of the infinitesimal operator A^ε (see ([6])) of the process X^ε, and $A^\varepsilon u^\varepsilon(x) = 0$. On the other hand, it is proved that the equation $A^\varepsilon v^\varepsilon(x) = 0$ with boundary conditions $v^\varepsilon(x)|_\Gamma = \psi(x)$ has a unique solution, which is given by formula (4). Thus, the function $u^\varepsilon(x)$ is the unique solution of problem (2) in the class of functions having bounded second derivatives in $D \setminus S$.

In what follows, for simplicity we shall assume that the set S is a domain formed by the intersection of the domain D with some $(n-1)$-dimensional plane of the space R^n. By $H(x)$ we denote the characteristic of the equation

$$\sum_{i=1}^n \tilde{b}_i(x) \frac{\partial v}{\partial x^i} = 0, \tag{5}$$

passing through the point $x \in D$.

We assume that, starting from any point $x \in D$, the characteristic $H(x)$ exits to the set S. By $H_1(x)$ we denote the point where the characteristic exits to the set S. The case when $H(x)$ exits to Γ is simpler and may be considered analogously.

In what follows we suppose that the domain S lies in the plane $x^1 = 0$, and we denote

$$\lim_{x^1 \downarrow 0} f(x^1, x) = f^+(x), \qquad \lim_{x^1 \uparrow 0} f(x^1, x) = f^-(x).$$

By χ_+ and χ_- we shall denote the characteristic functions of the sets $\{x^1 > 0\}$ and $\{x^1 < 0\}$, respectively.

The following lemmas play an essential role in the proofs of Theorems 2–5.

Denote by y_t^ε the first component of the process x_t^ε.

Lemma 1. For any $T > 0$, $x \in S$,

$$P_x \left\{ \lim_{\varepsilon \to 0} \sup_{u < T} |y_u^\varepsilon| = 0 \right\} = 1.$$

With the help of Lemma 1, from the results of R. Z. Khasminskii ([8], Theorem 3.1), the following lemma is derived.

Lemma 2. Let

$$\Lambda_+^t = \int_0^t \chi_+(x_u^\varepsilon)\, du, \qquad \Lambda_-^t = \int_0^t \chi_-(x_u^\varepsilon)\, du.$$

For arbitrary $\delta_1, \delta_2 > 0$ and $x \in S$, there exists such a $\delta_3 > 0$ that for $t < \delta_3$

$$P_x \left\{ \lim_{\varepsilon \downarrow 0} \frac{\Lambda_+^t}{\Lambda_-^t} < \left| \frac{\tilde{b}_1^-(x)}{\tilde{b}_1^+(x)} \right| + \delta_1 \right\} > 1 - \delta_2; \qquad P_x \left\{ \lim_{\varepsilon \downarrow 0} \frac{\Lambda_+^t}{\Lambda_-^t} > \left| \frac{\tilde{b}_1^-(x)}{\tilde{b}_1^+(x)} \right| - \delta_1 \right\} > 1 - \delta_2.$$

In what follows we shall assume that $|\tilde{b}_1(x)| = 1$. This can always be achieved by dividing both sides of the differential equation by $|\tilde{b}_1(x)|$.

Theorem 2. Suppose that the characteristics of the equation

$$\sum_{i=2}^{n} [b_i^+(x) + b_i^-(x)] \frac{\partial v}{\partial x^i} = 0, \tag{6}$$

specified in S, go out onto the boundary of the domain S. Denote by $\hat{S}$ the set of all points of the boundary of the domain S into which characteristics enter. Then $\lim_{\varepsilon \to 0} u^\varepsilon(x) = u(x)$ exists, and $u(x) = u(H_1(x))$. For $x^1 = 0$ the function $u(x)$ satisfies equation (6) and the boundary condition $u(x)|_{\hat{S}} = \psi(x)$.

To prove Theorem 2, consider the integral equation

$$\bar{x}_t - x_0 = \int_0^t \bar{b}(x_u)\, du, \tag{7}$$

where $\bar{b}(x)$ is an $(n-1)$-dimensional vector whose coordinates are $\{\tilde{b}_2^+(x) + \tilde{b}_2^-(x), \tilde{b}_3^+(x) + \tilde{b}_3^-(x), \dots, \tilde{b}_n^+(x) + \tilde{b}_n^-(x)\}$. With the aid of the lemmas formulated above one can prove that $M|\bar{x}_t - x_t^\varepsilon|^2 \to 0$ as $\varepsilon \to 0$. Hence, using Kolmogorov' s inequality for the martingale $\int_0^t \sigma(x_u)\, d\xi_u$, we obtain that

$$P\left\{\sup_{u<T} |\bar{x}_u - x_u^\varepsilon| > \delta\right\} \to 0$$

as $\varepsilon \to 0$. From the last assertion, taking (4) into account, the assertion of the theorem follows.

Theorem 3. Suppose that $\tilde{b}_i^+(x) = \tilde{b}_i^-(x) = 0$ for $i \geq 2$, $x \in D$. Suppose, moreover, that $a_{ij}(x)$ are continuous for $i, j \geq 2$, $x \in D$. Then there exists $\lim_{\varepsilon\to 0} u^\varepsilon(x) = u(x)$. The function $u(x)$ is constant on the characteristics of equation (5) and, for $x^1 = 0$, is a solution of the following boundary-value problem* in the domain S:

$$\frac{1}{2}\sum_{i,j=2}^n a_{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i,j=2}^n (b_i^+(x) + b_i^-(x))\frac{\partial u}{\partial x^i} = 0, \tag{8}$$

$$u\big|_{\bar{S}\setminus S} = \psi(x).$$

The proof of Theorem 3 is carried out analogously to the proof of Theorem 2. Instead of equation (7), one should consider the stochastic equation of the Markov family of functions governed by equation (8).

The following theorems answer the question of the limiting behavior of the solution of problem (2) for the equation $L_1^\varepsilon v^\varepsilon(x) = L^\varepsilon v^\varepsilon(x) + c(x)v^\varepsilon(x) = 0$, where $c(x) \leq 0$ in the domain D.

Theorem 4. Let $c(x)$ be continuous for $x \in D$. If the conditions of Theorem 2 are fulfilled, then $v^\varepsilon(x) \to v(x) = v_1(x) \cdot v_2(x)$ as $\varepsilon \to 0$. The function $v_1(x)$ can be found as the solution of the following boundary-value problem

$$\sum_{i=0}^n \tilde{b}_i(x)\frac{\partial v_0}{\partial x^i} + c(x)v_1(x) = 0, \qquad v_1(x)\big|_{\bar{S}} = 1. \tag{9}$$

The function $v_2(x)$ is constant on the characteristics of equation (5) and, for $x^1 = 0$, satisfies equation (10) and the boundary condition (10′):

$$\sum_{i=2}^{n}(\tilde{b}_i^+(x)+\tilde{b}_i^-(x))\frac{\partial v^2}{\partial x^i}+c(x)v_2(x)=0, \tag{10}$$

$$v_2(x)\big|_{\hat{S}}=\psi(x), \tag{10'}$$

where $\hat{S}$ is the set of those points of the boundary of the domain S into which characteristics enter.

In the next theorem we shall assume, for simplicity, that the functions $c(x)$, $\partial c/\partial x^1$, $|\tilde{b}_1(x)/a_{11}(x)|$ are defined and continuous in the domain D.

* We assume that $|\tilde{b}_1(x)|\equiv 1$.

Let $E=\{x\in S : c(x)<0\}$. Denote the boundary of the set $E\subset S$ by $\tilde{\Gamma}$.

Theorem 5. *Suppose that the conditions of Theorem 3 are satisfied. Then $v^\varepsilon(x)\to v(x)$ as $\varepsilon\to 0$. If $H_1(x)\in E\cup\tilde{\Gamma}$, then $v(x)=0$. If $H_1(x)\in\bar{E}$, then $v(x)=v_1(x)v_2(x)$, where $v_1(x)$ is the solution of problem (9), and the function $v_2(x)$ is constant on the characteristics of equation (5) and for $x^1=0$ is the solution of the following boundary-value problem:*

$$\frac{1}{2}\sum_{i,j=2}^{n}a_{ij}(x)\frac{\partial^2 v_2}{\partial x^i\partial x^j}+\sum_{i=2}^{n}(b_i^+(x)+b_i^-(x))\frac{\partial v_2}{\partial x^i}-\left|\frac{\partial c}{\partial x^1}\right|a_{11}(x)v_2(x)=0,$$

$$v_2(x)\big|_{\Gamma\setminus\tilde{\Gamma}}=\psi(x), \qquad v_2(x)\big|_{\tilde{\Gamma}}=0. \tag{11}$$

Note that $\left|\dfrac{\partial c(x)}{\partial x^1}\right|a_{11}(x)\geq 0$, so that equation (11) has a solution. The proof of Theorem 5 is carried out with the aid of a lemma analogous to Lemma 1 from (7), if one takes into account that the functions $v^\varepsilon(x)$ can be represented in the form

$$v^\varepsilon(x)=M_x^\varepsilon\psi(x^\varepsilon_{\tau^\varepsilon})\exp\left\{\int_0^{\tau^\varepsilon}c(x_u)\,du\right\}.$$

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Note: Figure translations are in progress. See original paper for figures.