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Abstract

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MATHEMATICS

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ON THE COMPOSITION OF PARAMETERS OF THE LORENTZ GROUP

(Presented by Academician V. A. Fock on 28 X 1961)

As shown in (^{1,2}), the matrix of a Lorentz transformation in Minkowski space $(x_4 = it,\ c = 1)$ can be represented in the form

$$L = L_q = |1+\mathbf{q}^2|^{-1}(1+\mathbf{q}_+)(1+\mathbf{q}_-^*) \qquad (1+\mathbf{q}^2 \neq 0). \tag{1}$$

Here $\mathbf{q} = \mathbf{a} + i\mathbf{b}$ $(\mathbf{q}^* = \mathbf{a} - i\mathbf{b})$ is a three-dimensional complex vector; $\mathbf{q}_\pm$ are four-dimensional antisymmetric matrices of the form

$$\mathbf{q}_\pm = \begin{pmatrix} \mathbf{q}^\times & \pm\mathbf{q} \\ \mp\mathbf{q} & 0 \end{pmatrix}, \qquad (\mathbf{q}^\times)_{ab} = \delta_{acb} q_c \quad (a, b, c = 1, 2, 3). \tag{2}$$

The relation between L and $\mathbf{q}$, determined by (1), (2), is one-to-one (^{1,2}). In view of this, $\mathbf{q}$ may be regarded as a complex vector-parameter of the proper orthochronous Lorentz group. The matrices $\mathbf{q}_\pm$ possess simple properties:

$$\mathbf{q}_\pm \mathbf{q}'_\mp = \mathbf{q}'_\mp \mathbf{q}_\pm, \qquad \mathrm{Sp}(\mathbf{q}_\pm \mathbf{q}'_\mp) = 0, \qquad \mathbf{q}_\pm \mathbf{q}'_\pm = -\mathbf{q}\mathbf{q}' + [\mathbf{q}\mathbf{q}']_\pm. \tag{3}$$

These relations make it possible to analyze very simply, in all details, the properties of the Lorentz matrix in the most general case. For example, with their aid it is not difficult to verify that the Lorentz matrix can be non-reducible to diagonal form (when $\mathbf{q}^2 = 0$).

For a given L, the vector $\mathbf{q}$ is uniquely found from the relation

$$\frac{1}{2}(\mathbf{q}_+ + \mathbf{q}_-^*) = \begin{pmatrix} -\mathbf{a}^\times & i\mathbf{b} \\ -i\mathbf{b} & 0 \end{pmatrix} = \frac{L - \widetilde{L}}{\mathrm{Sp}\, L}. \tag{4}$$

It follows from this that if L is a pure rotation

$$L = \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho = \rho_n = 1 + 2\frac{\mathbf{n}^\times + \mathbf{n}^{\times 2}}{1+\mathbf{n}^2}, \tag{5}$$

where $\mathbf{n}$ is a real vector (^{3}), then, according to (1)–(3), the relations $(\mathbf{q}-\mathbf{q}^*)/|1+\mathbf{q}^2| = [\mathbf{q}\mathbf{q}^*]/|1+\mathbf{q}^2| = 0$ must hold, whence either $\mathbf{q}-\mathbf{q}^* = 2i\mathbf{b} = 0$, or $[\mathbf{q}\mathbf{q}^*] = 2i[\mathbf{b}\mathbf{a}] = 0$, $|\mathbf{q}^2| = \infty$. In the latter case, introducing the unit real vector $\mathbf{q}_1 = \mathbf{q}/\sqrt{\mathbf{q}^2}$, we obtain $\rho = 2\mathbf{q}_1 \cdot \mathbf{q}_1 - 1$, which corresponds to a rotation by π about the direction $\mathbf{q}_1$. Thus, a rotation corresponds to a real vector $\mathbf{q}$, or to a *linear* vector $\mathbf{q}$ when $|\mathbf{q}^2| = \infty$. If, however, L corresponds to a transformation to a frame moving with velocity $\mathbf{v}$, then

$$L = \begin{pmatrix} 1 + (\gamma - 1)\dfrac{\mathbf{v}\cdot\mathbf{v}}{\mathbf{v}^2} & i\gamma\mathbf{v} \\ -i\gamma\mathbf{v} & \gamma \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}}. \tag{6}$$

Comparing this with (1), we obtain $\mathbf{q}+\mathbf{q}^* = 2\mathbf{a} = 0$, $\mathbf{q} = i\mathbf{u}$ $\left(\mathbf{u} = \dfrac{\mathbf{v}}{1+\sqrt{1-\mathbf{v}^2}},\ \mathbf{u}^2 < 1\right)$. Thus, in the case of motion without rotation, $\mathbf{q}$ must be a purely imaginary vector, with modulus less than unity. In all other cases $\mathbf{q}$ determines a Lorentz transformation that represents some combination of motion and rotation. To clarify this question in detail, it is necessary to know the law of composition of the vector parameters $\mathbf{q}$. If we denote $L_q L_{q'} = L(\mathbf{q}, \mathbf{q}')$, then with the aid of (1), (3) one can easily verify that

$$(\mathbf{q}, \mathbf{q}') = \frac{\mathbf{q} + \mathbf{q}' + [\mathbf{q}\mathbf{q}']}{1 - \mathbf{q}\mathbf{q}'}. \tag{7}$$

Let us note that the general law of composition obtained for the complex vector parameters of the Lorentz group differs from the corresponding law for the rotation group (3) only in that the vector $\mathbf{q}$ is complex. It is not difficult to verify the following properties of formula (7):

$$(\mathbf{q}, 0) = (0, \mathbf{q}) = \mathbf{q}, \qquad (\mathbf{q}, -\mathbf{q}) = (-\mathbf{q}, \mathbf{q}) = 0; \tag{8}$$

$$((\mathbf{q}, \mathbf{q}'), \mathbf{q}'') = (\mathbf{q}, (\mathbf{q}', \mathbf{q}'')), \qquad -(\mathbf{q}, \mathbf{q}') = (-\mathbf{q}', -\mathbf{q}); \tag{9}$$

$$(\mathbf{q}, \mathbf{q}')^2 = (\mathbf{q}', \mathbf{q})^2, \qquad (\mathbf{q}, \mathbf{q}') = (\rho_q \mathbf{q}', \mathbf{q}) = (\mathbf{q}', \rho_{-\mathbf{q}} \mathbf{q}). \tag{10}$$

In view of this, the set Q of all vectors $\mathbf{q}$ in three-dimensional complex space forms a continuous group with multiplication law (7), where $\mathbf{q} = 0$ is the identity element, and $(-\mathbf{q})$ is the element inverse to $\mathbf{q}$. In general, $(\mathbf{q}, \mathbf{q}') \neq (\mathbf{q}', \mathbf{q})$, and the equality $(\mathbf{q}, \mathbf{q}') = (\mathbf{q}', \mathbf{q})$ is possible only under the condition $[\mathbf{q}\mathbf{q}'] = 0$, i.e., when $\mathbf{q}$ and $\mathbf{q}'$ are linearly dependent. The set Q_0 of vectors $\mathbf{q}$ satisfying the

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* We call linear a three-dimensional complex vector $\mathbf{q}$ satisfying the condition $[\mathbf{q}, \mathbf{q}^*] = 0$ (see (4)).

condition $1+\mathbf{q}^2 \neq 0$, obviously isomorphic to the proper orthochronous Lorentz group, is a subgroup of the set Q. Indeed, $1+(\mathbf{q},\mathbf{q}')^2 = (1+\mathbf{q}^2)(1+\mathbf{q}'^2)/(1-\mathbf{q}\mathbf{q}')^2$, and therefore, if $1+\mathbf{q}^2 \neq 0$, $1+\mathbf{q}'^2 \neq 0$, then also $1+(\mathbf{q},\mathbf{q}')^2 \neq 0$.

As is known, the general Lorentz transformation is usually defined as the product of transformations corresponding to a rotation and to inertial motion of the system. Accordingly, the complex $\mathbf{q}$ can in general be represented in the following way:

$$\mathbf{q} = (\mathbf{n}, i\mathbf{u}), \qquad \mathbf{q}^* = (\mathbf{n}, -i\mathbf{u}). \tag{11}$$

With the aid of (8), (9), from this we find $\mathbf{n}$ and $\mathbf{u}$ without difficulty. Indeed, $\mathbf{n} = (\mathbf{q}, -i\mathbf{u}) = (\mathbf{q}^*, i\mathbf{u})$, whence $(-\mathbf{q}^*, \mathbf{q}) = (i\mathbf{u}, i\mathbf{u}) = \dfrac{2i\mathbf{u}}{1+\mathbf{u}^2}$. Thus,

$$\mathbf{v} = \frac{2\mathbf{u}}{1+\mathbf{u}^2} = i(-\mathbf{q}, \mathbf{q}^*) = 2\,\frac{\mathbf{b}-[\mathbf{ab}]}{1+\mathbf{a}^2+\mathbf{b}^2}. \tag{12}$$

Hence

$$1-\mathbf{v}^2 = 1+(-\mathbf{q},\mathbf{q}^*)^2 = \frac{|1+\mathbf{q}^2|^2}{(1+|\mathbf{q}|^2)^2} = \frac{1}{L_{44}^2}. \tag{13}$$

Consequently, $1+\mathbf{q}^2 = 0$ corresponds to $|\mathbf{v}| = 1$, i.e., to motion of the system with the speed of light. For $\mathbf{n}$ we obtain

$$\mathbf{n} = \frac{\sqrt{1+\mathbf{q}^2}\,\mathbf{q}^* + \left(\sqrt{1+\mathbf{q}^2}\right)^*\mathbf{q}}{\sqrt{1+\mathbf{q}^2} + \left(\sqrt{1+\mathbf{q}^2}\right)^*}. \tag{14}$$

In the same way one can easily obtain the representation $\mathbf{q} = (i\mathbf{u}', \mathbf{n}')$, which differs in the order of the spatial rotation and the motion. In this case

$$\mathbf{v}' = i(\mathbf{q}^*, -\mathbf{q}) = \rho_n\mathbf{v} = 2\,\frac{\mathbf{b}+[\mathbf{ab}]}{1+\mathbf{a}^2+\mathbf{b}^2}, \qquad \mathbf{n}' = \mathbf{n}, \tag{15}$$

Consider the case $\mathbf{q}^2 = \mathbf{q}^{*2}$, $1+\mathbf{q}^2 < 0$ ($\mathbf{ab} = 0$, $1+\mathbf{a}^2-\mathbf{b}^2 < 0$). In this case, according to (14), $\mathbf{n} \parallel \mathbf{b}$, $|\mathbf{n}| = \infty$, which corresponds to a rotation by π about the axis $\mathbf{b}$. Hence, in particular, it follows that if $\mathbf{q} = i\mathbf{b}$, $|\mathbf{b}| > 1$, then such a vector-parameter determines a Lorentz transformation consisting of a motion with velocity $\mathbf{v} = \dfrac{2\mathbf{b}}{1+\mathbf{b}^2}$ (13) and a rotation by π about the direction of the velocity. Similarly, it is not difficult to verify that for $|\mathbf{q}^2| = \infty$, $[\mathbf{q}\mathbf{q}^*] \neq 0$, the corresponding Lorentz transformation consists of a motion with velocity $\mathbf{v} = -\dfrac{2[\mathbf{ab}]}{\mathbf{a}^2 + \mathbf{b}^2}$ and a subsequent rotation by π about the axis $\mathbf{n}$ ($\mathbf{n} \perp \mathbf{v}$), coinciding with the real part of the vector $\mathbf{q}_1 = \mathbf{q}/\sqrt{\mathbf{q}^2}$.

Relations (1)–(3), (7) make it possible to clarify easily a number of properties of Lorentz matrices. Since $L^+_{\mathbf{q}} = L_{-\mathbf{q}^*}$, we have $L^+_{\mathbf{q}} L_{\mathbf{q}} = L_{i\mathbf{v}}$ and $\mathrm{Sp}(L^+_{\mathbf{q}} L_{\mathbf{q}}) = \dfrac{4}{1-\mathbf{v}^2}$. Hence it follows, in particular, that the Lorentz group is unbounded, since $1 - \mathbf{v}^2$ can be arbitrarily close to zero. The matrix of a proper Lorentz transformation is normal ($L^+_{\mathbf{q}} L_{\mathbf{q}} = L_{\mathbf{q}} L^+_{\mathbf{q}}$) if $(-\mathbf{q}^*, \mathbf{q}) = (\mathbf{q}, -\mathbf{q}^*)$, i.e., if $[\mathbf{qq}^*] = 0$. For an improper transformation $L'_{\mathbf{q}} = \pm P L_{\mathbf{q}}$ (P is the spatial-inversion transformation $P L_{\mathbf{q}} = L_{\mathbf{q}^*} P$), therefore $L'^+_{\mathbf{q}} L'_{\mathbf{q}} = L^+_{\mathbf{q}} L_{\mathbf{q}} = L_{(-\mathbf{q}^*, \mathbf{q})}$, $L'_{\mathbf{q}} L'^+_{\mathbf{q}} = P L_{(\mathbf{q}, -\mathbf{q}^*)}$, $P = L_{(\mathbf{q}^*, -\mathbf{q})}$. Consequently, $L'_{\mathbf{q}}$ is normal if $(-\mathbf{q}^*, \mathbf{q}) = (\mathbf{q}^*, -\mathbf{q})$, i.e. $\mathbf{q} = \mathbf{q}^*$; in this case it is also unitary. For a purely imaginary vector-parameter ($\mathbf{q}^* = -\mathbf{q}$), $L'^2_{\mathbf{q}} = P L_{\mathbf{q}} P L_{\mathbf{q}} = L_{\mathbf{q}^*} L_{\mathbf{q}} = 1$.

Relations (4), (12), (14) in the most general case answer the question of what velocity and rotation correspond to an arbitrarily given Lorentz matrix. Relations (11), (7), (1), (2) make it possible to solve the inverse problem, i.e., to obtain an explicit expression for the Lorentz matrix corresponding to a given rotation and motion of the reference frame. In doing so, we may, if desired, either combine the velocities and rotations by including them in a single complex vector $\mathbf{q}$, or consider them separately. Obviously, from the physical point of view, (7) represents a peculiar form of the general relativistic law of composition of arbitrary velocities and rotations. In particular, adding two pure motions with parameters $\mathbf{q} = i\mathbf{u}$, $\mathbf{q}' = i\mathbf{u}'$ ($\mathbf{u}^2 < 1$, $\mathbf{u}'^2 < 1$), we obtain, according to (7),

$$\mathbf{q}'' = (i\mathbf{u},\, i\mathbf{u}') = \frac{i(\mathbf{u} + \mathbf{u}') - [\mathbf{uu}']}{1 + \mathbf{uu}'}. \tag{16}$$

Since $\mathbf{q}''$ is not purely imaginary, in the general case the superposition of pure motions is associated with a certain rotation. Formula (12) gives, in this case,

$$\mathbf{v}'' = 2\,\frac{\mathbf{u}(1 - \mathbf{u}'^2) + \mathbf{u}'(1 + \mathbf{u}^2) + 2\mathbf{u}'\mathbf{uu}'}{(1 + \mathbf{u}^2)(1 + \mathbf{u}'^2) + 4\mathbf{uu}'}.$$

Passing here from $\mathbf{u}, \mathbf{u}'$ to $\mathbf{v}, \mathbf{v}'$, we obtain

$$\mathbf{v}'' = \frac{\mathbf{v}\sqrt{1 - \mathbf{v}^2} + \mathbf{v}'\left[1 + \dfrac{\mathbf{vv}'}{\mathbf{v}'^2}\left(1 - \sqrt{1 - \mathbf{v}'^2}\right)\right]}{1 + \mathbf{vv}'}, \tag{17}$$

which coincides with the well-known general formula for the composition of velocities (^{5}). Moreover, according to (15), in the present case the motion with velocity $\mathbf{v}''$ will be accompanied by a rotation of the reference frame determined by the vector $\mathbf{n} = -[\mathbf{uu}']/(1 + \mathbf{uu}')$. The axis of this rotation is perpendicular to the plane of the velocity vectors $\mathbf{v}, \mathbf{v}'$.

Let us note in conclusion that, when the proposed parametrization is used, some results of the theory of representations of the Lorentz group become obvious without any computations. Thus, if the six parameters $\mathbf{q}_a$, $\mathbf{q}_b^*$ correspond to the infinitesimal operators of the representations I_a, I_b', then it is immediately clear that each I_a will commute with each I_b', since,

according to (7), $\mathbf{q}$ and $\mathbf{q}^*$ are composed independently of one another. Since the composition law (7) coincides with that for the rotation group (3), the commutation rules for I_a among themselves (as well as for I_b') will be the same as for the rotation group. These rules, $I_a I_b - I_b I_a = \delta_{abc} I_c$, in accordance with the general theory of Lie groups, follow directly from the fact that the second-order terms in formula (7) have the form $[\mathbf{qq}']_c = \delta_{abc} q_a q_b'$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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