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A. B. VASIL'EVA

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Abstract

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A. B. VASIL' EVA

ASYMPTOTIC FORMULAS FOR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE, VALID ON A SEMI-INFINITE INTERVAL

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In papers ([1,2]) asymptotic formulas (with respect to the small parameter μ) were obtained for the solution of the system of differential equations

$$\begin{gathered} \mu\frac{dz}{dt} = F(z,y,t), \qquad \mu \geqslant 0, \\ \frac{dy}{dt} = f(z,y,t), \\ z\,\Big|_{t=0} = z^0, \qquad y\,\Big|_{t=0} = y^0, \end{gathered} \tag{1}$$

valid on the finite interval of variation of t: $0 \leqslant t \leqslant T$. In the present note it will be shown that, under certain additional restrictions on system (1), these formulas also prove to be valid on the semi-infinite interval $0 \leqslant t < \infty$.

We shall restrict ourselves to the case where z and y are scalar quantities. The limiting passage $\mu \to 0$ for problem (1) was considered earlier in paper ([5]). We, however, shall proceed from the results of paper ([3]), where the limiting passage on the finite interval $0 \leqslant t \leqslant T$ was proved; by adding to the conditions of theorem ([3]) certain additional restrictions, we shall first extend result ([3]) to the interval $0 \leqslant t < \infty$, and then, slightly modifying the proof of ([2]), obtain on this interval the required asymptotic formulas.

Putting $\mu = 0$ in the first equation of (1), we obtain the equation

$$F(z,y,t) = 0. \tag{2}$$

Suppose that this equation has, in a certain domain D_{tyz}, bounded with respect to y and z and semi-infinite with respect to t: $0 \leqslant t < \infty$, $y_1(t) \leqslant y \leqslant y_2(t)$, $z_1(y,t) \leqslant z \leqslant z_2(y,t)$ (y_1, y_2, z_1, z_2 are continuous and bounded functions), a unique solution (root) $z = \varphi(y,t)$. We shall assume that the functions

$F(z,y,t)$ and $f(z,y,t)$, together with their partial derivatives up to the second order, are continuous and bounded in this domain.

Let the root $z=\varphi(y,t)$ be stable in the sense of ([3,4]). This means that

$$\frac{\partial F}{\partial z}(\varphi(y,t),y,t) < -\alpha \quad (0<\alpha=\text{const})$$

in D_{ty} $(y_1(t)\leqslant y\leqslant y_2(t),\ 0\leqslant t<\infty)$. Suppose that the initial point $t=0,\ y=y^0,\ z=z^0$ belongs to D_{tyz}.

Substituting $z=\varphi(y,t)$ into the second equation of (1), we obtain

$$\frac{dy}{dt}=f(\varphi(y,t),y,t)=\tilde{f}(y,t). \tag{3}$$

Denote by $\bar{y}(t)$ the solution of this equation satisfying the initial condition $\bar{y}\,\big|_{t=0}=y^0$. Suppose that the curve $y=\bar{y}(t)$ pri-

belongs to D_{ty} for $0\le t<\infty$. Suppose, finally, that

$$\frac{\partial \tilde{f}}{\partial y}(\bar{y}(t),t) < -a \qquad (0<a=\text{const}) \tag{4}$$

for $t\ge T$, where T is some constant which may be arbitrarily large. Condition (4) ensures the asymptotic stability of the solution $\bar{y}(t)$ of equation (3) and in the present case replaces the requirement, used in ([3,4]) for the case of a finite interval, of continuous dependence of the solution of equation (3) on changes in the right-hand side and in the initial values.

Under these assumptions, by arguments analogous to those used in ([4]) (in ([4]) a certain special case is described, which includes the case considered here), the existence of the limit is proved:

$$\begin{aligned} &\lim_{\mu\to 0} z(t,\mu)=\varphi(\bar{y}(t),t)=\bar{z}(t), && 0<t<\infty;\\ &\lim_{\mu\to 0} y(t,\mu)=\bar{y}(t), && 0\le t<\infty. \end{aligned} \tag{5}$$

It can further be shown that, under certain additional smoothness conditions on the right-hand sides of (1), the asymptotic formulas indicated in ([1,2]) are valid for the solution of this system. We shall use the same formal constructions as in ([1,2]), namely, after first making in (1) the change of variables $\tau=t/\mu$, we construct a formal solution of this system in the form of an expansion in powers of μ with coefficients depending on τ:

$$x={}^{(1)}x_0(\tau)+\mu\,{}^{(1)}x_1(\tau)+\cdots \tag{6}$$

(x denotes y and z together). Next we construct formal solutions of system (1) in the form of an expansion in μ with coefficients depending on t, and in the form of an expansion in μ and t:

$$x = {}^{(2)}x_0(t) + \mu\, {}^{(2)}x_1(t) + \cdots; \tag{7}$$

$$x = {}^{(2)}x_{00} + \mu\, {}^{(2)}x_{01} + t\, {}^{(2)}x_{10} + \cdots. \tag{8}$$

The coefficients in (6) are determined from the corresponding variational equations and satisfy the additional conditions ${}^{(1)}x_0\big|_{t=0} = x^0$, ${}^{(1)}x_i\big|_{t=0} = 0$. The coefficients in (7) are also determined from variational equations, which in this case will all be first-order differential equations, and satisfy the additional conditions

$$ {}^{(2)}y_k\big|_{t=0} = \frac{(-1)^k}{k!} \int_0^\infty \tau^k \frac{d^k}{d\tau^k}\, {}^{(1)}f_{k-1}\, d\tau, \tag{9}$$

where ${}^{(1)}f_{k-1}$ is the $(k-1)$-st coefficient of the expansion of the function $f(z, y, \tau\mu)$ of type (6). Form the expressions

$$X_n = \left({}^{(1)}x_0 + \cdots + \mu^{n\,(1)}x_n\right) + \left({}^{(2)}x_0 + \cdots + \mu^{n\,(2)}x_n\right) - \left({}^{(2)}x_{00} + \cdots + \mu^{n\,(2)}x_{0n} + \cdots + t^{n\,(2)}x_{n0}\right). \tag{10}$$

Suppose that, in addition to the conditions stated above, the right-hand sides of system (1) possess continuous partial derivatives up to order $(n+1)$ inclusive in a neighborhood of the curve $t = 0$, $y = y^0$, $z = {}^{(1)}z_0(\tau)$, $(0 \le \tau < \infty)$; $0 < t < \infty$, $y = \bar{y}(t)$, $z = \bar{z}(t)$ (this curve is $\cdots$).

is a limiting curve as $\mu \to 0$ for the integral curve under consideration of system (1)), the neighborhood may be chosen arbitrarily small, but remains fixed as $\mu \to 0$; suppose, in addition, that the partial derivatives up to order n and $n-1$, respectively, of the functions $F(z, y, t)$ and $f(z, y, t)$ are differentiable $n+1$ times with respect to z at the points of the vertical

$$t = 0, \quad y = y^0, \quad z = \overset{(1)}{z}_0(\tau) \quad (0 \le \tau < \infty).$$

Then, for the solution $x(t, \mu)$ of system (1), the relation

$$x(t, \mu) = X_n + R_{n+1}, \tag{11}$$

holds, where $|R_{n+1}| < c\mu^{n+1}$, and c is independent of μ and t for sufficiently small μ ($\mu \le \mu^0$) and $0 \le t < \infty$, i.e., the asymptotic formula given in (1, 2) for the finite interval $0 \le t \le T$ turns out to be valid on a semi-infinite interval.

However, expression (10) for X_n becomes inconvenient for computations for large t, because of the presence of terms of order t^k ($k = 1, \dots, n$), which are contained in the first and third of the sums enclosed in parentheses, so that the entire expression (10) for large t has a form containing an indeterminacy of the type $\infty - \infty$. Let us write (10) in a more convenient form.

First consider the formula of the first approximation. We represent (10) for $n = 1$ as follows:

$$X_1 = \overset{(2)}{x}_0 + \mu \overset{(2)}{x}_1 + \Pi_0(x) + \Pi_1(x). \tag{12}$$

Here the following notation has been introduced:

$$\Pi_0(w) = \overset{(1)}{w}_0 - \overset{(2)}{w}_{00}, \qquad \Pi_1(w) = \mu \overset{(1)}{w}_1 - (\mu \overset{(2)}{w}_{01} + t \overset{(2)}{w}_{10}),$$

(w is an arbitrary differentiable function of z, y, t). By direct computation it is not difficult to obtain

$$\Pi_0(z) = \overset{(1)}{z}_0 - \overset{(2)}{z}_{00}, \qquad \Pi_0(y) = \overset{(1)}{y}_0 - \overset{(2)}{y}_{00} = y^0 - y^0 \equiv 0,$$

$$\Pi_1(y) = -\mu \int_\tau^\infty \Pi_0(v)\, d\tau = -\mu \int_\tau^\infty (\overset{(1)}{v}_0 - \overset{(2)}{v}_{00})\, d\tau = -\mu \int_\tau^\infty \Pi_0(v)\, d\tau,$$

while $\Pi_1(z)$ satisfies the linear equation

$$\frac{d}{d\tau}\Pi_1(z) = \overset{(1)}{w}_{z0}\Pi_1(z) + \Pi_0(w_z)(\mu \overset{(2)}{z}_{01} + t \overset{(2)}{z}_{10}) +$$

$$+ \overset{(1)}{w}_{y0}\Pi_1(y) + \Pi_0(w_y)(\mu \overset{(2)}{y}_{01} + t \overset{(2)}{y}_{10}),$$

in which the nonhomogeneity may be regarded as known, and with the initial condition

$$\Pi_1(z)\,\Big|_{t=0} = -\mu \overset{(2)}{z}_{01}.$$

$\Pi_1(y)$ is easily majorized by a quantity $c\mu e^{-\beta t/\mu}$ (c and β are constants independent of t and μ), and the same estimate holds for $\Pi_1(z)$, as is not difficult to verify from the elementary integral representation for $\Pi_1(z)$.

In a form analogous to (12), one can also write the formula of the n-th approximation:

$$X_n = \overset{(2)}{x}_0 + \dots + \mu^n \overset{(2)}{x}_n + \Pi_0(x) + \dots + \Pi_n(x), \tag{13}$$

where

$$\Pi_k(x) = \mu^k \overset{(1)}{x}_k - (\mu^k \overset{(2)}{x}_{0k} + \mu^{k-1} t \overset{(2)}{x}_{1,k-1} + \dots + t^k \overset{(2)}{x}_{k0}).$$

Here, just as in the case of the first approximation,

$$\Pi_k(y) = -\mu \int_\tau^\infty \Pi_{k-1}(v)\, d\tau,$$

$$\frac{d}{d\tau}\Pi_k(z) = \overset{(1)}{w}_{z0}\Pi_k(z) + \Phi_k,$$

$$\Pi_k(z)\,\Big|_{t=0} = -\mu^k \overset{(2)}{z}_{0k},$$

where Φ_k contains $\Pi_k(y)$ and $\Pi_i(x)$ $(i < k)$. For $\Pi_k(x)$ the estimate

$$|\Pi_k(x)| < c\mu^k e^{-\beta t/\mu}. \tag{14}$$

holds.

The proof of (11) for $0 \le t < \infty$ is based on three facts: the validity of (11) for $0 \le t \le T$, proved in the author' s previous works ([1,2]), the estimate (14), and the limiting equalities $\lim_{\mu\to 0} x_{\mu k}(t,\mu) = k!\,x_k^{(2)}(t)$, proved earlier ([6]) for $t \le T$, and, by virtue of condition (4), also valid for $t \ge T$, as can be verified by using methods analogous to those employed in the preceding works ([6,1,2]).

Remark. In paper ([7]), asymptotic formulas were given for the case when the system under consideration contains two small parameters and has the form

$$\begin{aligned}
\mu_1\mu_2\,\frac{dz}{dt} &= w(z,y,x,t),\\
\mu_2\,\frac{dy}{dt} &= v(z,y,x,t),\\
\frac{dx}{dt} &= u(z,y,x,t),\\
z|_{t=0} = z^0, \qquad y|_{t=0} &= y^0, \qquad x|_{t=0} = x^0.
\end{aligned} \tag{15}$$

In the formula presented in ([7]), even in the case of a finite interval of variation of t, there occurs $\infty - \infty$ (the difference of terms of order $\tau_k^2 = (t/\mu_2)^k$). To eliminate this formal inconvenience, one may write the indicated formula also in the form of (12), (13), i.e., as a partial sum of the formal expansion of (15) in powers of μ_1 and μ_2, with exponentially small corrections majorized by the quantities $c\mu_1^k\mu_2^l e^{-\beta t/\mu_1\mu_2}$ and $c\mu_1^k\mu_2^l e^{-\beta t/\mu_2}$, and representable in a form similar to that in which $\Pi_k(y)$, $\Pi_k(z)$ were represented above.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.