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Abstract

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Boundary Value Problems for an Equation in Hilbert Space

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In the present article we consider boundary value problems for a second-order differential equation in Hilbert space. The reason for this consideration was provided by certain problems in the theory of periodic waveguides, studied in the work of M. G. Krein and G. Ya. Lyubarskii (1).

Let the differential equation

$$d^2u/dt^2 - Au + \lambda B(t)u = 0, \tag{1}$$

be given on the interval $[0, T]$, where $u(t)$ is the unknown function with values in a Hilbert space H; A is a self-adjoint positive definite operator in H, having a completely continuous inverse; $B(t)$ is a bounded self-adjoint positive definite operator in H, depending sufficiently smoothly on t; λ is a parameter.

For equation (1) we consider a boundary value problem, i.e. the problem of finding a solution satisfying a system of boundary conditions of the form

$$\begin{aligned} \alpha_{11}u(0) + \alpha_{12}u'(0) + \beta_{11}u(T) + \beta_{12}u'(T) = 0, \\ \alpha_{21}u(0) + \alpha_{22}u'(0) + \beta_{21}u(T) + \beta_{22}u'(T) = 0. \end{aligned} \tag{2}$$

Equation (1) can be reduced to a system of first-order equations:

$$du/dt = A^{1/2}v,$$

$$dv/dt = A^{1/2}u - \lambda A^{-1/2}B(t)u.$$

We make a change of the unknown functions: $z = u - v$, $w = u + v$. For z and w we obtain the system:

$$dz/dt = -A^{1/2}z + f, \tag{3}$$

$$dw/dt = A^{1/2}w - f, \tag{4}$$

where $f(t) = \lambda A^{-1/2} B(t) u$.

For equation (3) the problem of finding the solution from its initial value $z(0)$ is correct. This solution can be written in the form (2)

$$z(t) = e^{-A^{1/2} t} z(0) + \int_0^t e^{-A^{1/2}(t-\tau)} f(\tau)\, d\tau. \tag{5}$$

For equation (4) the problem of finding the solution from its terminal value $w(T)$ is correct. This solution is given by the formula

$$w(t) = e^{-A^{1/2}(T-t)} w(T) + \int_t^T e^{-A^{1/2}(\tau-t)} f(\tau)\, d\tau. \tag{6}$$

From (5) and (6) we obtain an integral equation for the solution of equation (1)

$$2u(t) = e^{-A^{1/2} t} z(0) + e^{-A^{1/2}(T-t)} w(T) + \int_0^T e^{-A^{1/2}|t-\tau|} f(\tau)\, d\tau, \tag{7}$$

where $z(0) = u(0) - A^{-1/2} u'(0)$, $w(T) = u(T) + A^{-1/2} u'(T)$, and $f(t) = \lambda A^{-1/2} B(t) u(t)$.

Putting $t = T$ in equation (5), and $t = 0$ in equation (6), we obtain two additional relations:

$$\begin{aligned} z(T) &= e^{-A^{1/2} T} z(0) + \int_0^T e^{-A^{1/2}(T-\tau)} f(\tau)\, d\tau, \\ w(0) &= e^{-A^{1/2} T} w(T) + \int_0^T e^{-A^{1/2} \tau} f(\tau)\, d\tau. \end{aligned} \tag{8}$$

Equations (2) and (8) may be regarded as a nonhomogeneous linear system of equations with the unknown elements $u(0)$, $u'(0)$, $u(T)$, and $u'(T)$:

$$\begin{aligned} \alpha_{11} u(0) + \alpha_{12} u'(0) + \beta_{11} u(T) + \beta_{12} u'(T) &= f_1, \\ \alpha_{21} u(0) + \alpha_{22} u'(0) + \beta_{21} u(T) + \beta_{22} u'(T) &= f_2, \\ -Qu(0) + PQu'(0) + u(T) - Pu'(T) &= f_3, \\ u(0) + Pu'(0) - Qu(T) - PQu'(T) &= f_4, \end{aligned} \tag{9}$$

where $P = A^{-1/2}$ and $Q = e^{-A^{1/2} T}$.

The determinant of this system is a self-adjoint operator D, possessing the following properties:

1. The inverse operator D^{-1} to the operator D exists if and only if the number $\lambda = 0$ is not an eigenvalue for problem (1)–(2).
2. In order that the operator D^{-1} be bounded, it is necessary and sufficient that $\alpha_{12}\beta_{22} - \beta_{12}\alpha_{22} \neq 0$.
3. If the operator D^{-1} is unbounded, then it can be represented in the form $C_1 A^{1/2}$, or $C_2 A$, or $C_3 A^{1/2} e^{A^{1/2}T}$, where C_1, C_2, C_3 are bounded operators.

The boundary conditions (2) are called self-adjoint if

$$\begin{aligned} \alpha_{11}\overline{\alpha}_{12} - \alpha_{12}\overline{\alpha}_{11} &= \beta_{11}\overline{\beta}_{12} - \beta_{12}\overline{\beta}_{11}, \\ \alpha_{11}\overline{\alpha}_{22} - \alpha_{12}\overline{\alpha}_{21} &= \beta_{11}\overline{\beta}_{22} - \beta_{12}\overline{\beta}_{21}, \\ \alpha_{21}\overline{\alpha}_{22} - \alpha_{22}\overline{\alpha}_{21} &= \beta_{21}\overline{\beta}_{22} - \beta_{22}\overline{\beta}_{21}. \end{aligned} \tag{10}$$

As is known (see ([4])), in the scalar case these conditions are necessary and sufficient for the self-adjointness of the operator d^2u/dt^2.

Lemma 1. *Under self-adjoint boundary conditions, the elements $u(0)$, $u(T)$, $A^{-1/2}u'(0)$, and $A^{-1/2}u'(T)$, found from system (9), are expressed by means of bounded operators through the elements f_1, f_2, f_3, f_4.*

If the elements $z(0)$ and $w(T)$ found from (2) and (8) are substituted into (7), then we arrive at an integral equation for the solution $u(t)$ of problem (1)–(2):

$$u(t) = \lambda \int_0^T R(t,\tau)B(\tau)u(\tau)\,d\tau, \tag{11}$$

where $R(t,\tau)$ is a certain function of the operator A. By the substitution $B^{1/2}(t)u(t) = y(t)$, equation (11) is reduced to the equation

$$y(t) = \lambda \int_0^T B^{1/2}(t)R(t,\tau)B^{1/2}(\tau)y(\tau)\,d\tau. \tag{12}$$

Under self-adjoint boundary conditions the kernel $R(t,\tau)$ is Hermitian: $R(\tau,t) = R^*(t,\tau)$, where the asterisk denotes passage to the adjoint operator for fixed t and τ.

Lemma 2. *Let $\mathcal{K}(t,\tau)$, for almost all (t,τ) in the square $0 \le t \le T$, $0 \le \tau \le T$, be a linear completely continuous operator acting in the Hilbert space H. If the function $\mathcal{K}(t,\tau)$ is integrable*

by Bochner, and

$$\int_0^T \int_0^T \|\mathcal{K}(t,\tau)\|^2\,dt\,d\tau < \infty,$$

then the linear operator

$$Ku = \int_0^T \mathcal{K}(t,\tau)u(\tau)\,d\tau$$

is completely continuous in the space $L_2(H,[0,T])$.

By the space $L_2(H,[0,T])$ is meant the totality of all Bochner-integrable functions $u(t)$ with values in H for which

$$\|u\|_{L_2}^2 = \int_0^T \|u(t)\|^2\,dt < \infty.$$

Theorem 1. If, under the self-adjoint boundary conditions (2), the number $\lambda = 0$ is not an eigenvalue of problem (1)–(2), then the kernel of equation (12) satisfies the conditions of Lemma 2 and, consequently, problem (1)–(2) reduces to the problem of eigenfunctions of a completely continuous self-adjoint operator in the Hilbert space $L_2(H,[0,T])$.

Let us give examples of integral equations to which various boundary-value problems for equation (1) are reduced.

1st boundary-value problem:

$$u(0) = u(T) = 0.$$

The integral equation has the form

$$y(t) = \frac{\lambda}{2}\int_0^T B^{1/2}(t)\{S(|t-\tau|) - [I - S(2T)]^{-1}[S(t+\tau) + S(2T-t-\tau) - S(2T+t-\tau) - S(2T-t+\tau)]\}A^{-1/2}B^{1/2}(\tau)y(\tau)\,d\tau, \tag{13}$$

where $y(t) = B^{1/2}(t)u(t)$, and $S(t)$ is the semigroup $e^{-A^{1/2}t}$.

2nd boundary-value problem:

$$u'(0) = u'(T) = 0.$$

The equation has the form

$$y(t) = \frac{\lambda}{2}\int_0^T B^{1/2}(t)\{S(|t-\tau|) + [I - S(2T)]^{-1}[S(t+\tau) + S(2T-t-\tau) + S(2T+t-\tau) + S(2T-t+\tau)]\}A^{-1/2}B^{1/2}(\tau)y(\tau)\,d\tau \tag{14}$$

(with the same notation).

The difference of the integral operators standing on the right in (14) and (13) is a positive operator. If we denote by $\lambda_n^{(1)}$ the eigenvalues of the first boundary-value problem, and by $\lambda_n^{(2)}$ those of the second boundary-value problem, then it follows from the preceding that $\lambda_n^{(1)} \geq \lambda_n^{(2)}$.

The simplest example of a partial differential equation to which the preceding results are applicable is the equation

$$\partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2 + \cdots + \partial^2 u/\partial x_n^2 + \omega^2/c^2(x_1, x_2, \dots, x_n)u = 0, \tag{15}$$

considered in the cylindrical domain $[0, T] \times G$, where G is a bounded domain in the $(n-1)$-dimensional space $(x_2, \dots, x_n)$ with sufficiently smooth boundary. We impose in the domain G homogeneous boundary conditions under which the operator $\partial^2 u/\partial x_2^2 + \cdots + \partial^2 u/\partial x_n^2$ is self-adjoint and negative definite. For example, let $u = 0$ on the boundary ΓG of the domain G.

Introducing the notation

$$x_1 = t, \qquad \partial^2 u/\partial x_2^2 + \cdots + \partial^2 u/\partial x_n^2 = -Au, \qquad \frac{1}{c^2}u = B(t)u$$

and $\omega^2 = \lambda$, we arrive at equation (1).

In the theory of cylindrical waveguides (see ([1])) there arise problems of finding solutions of equation (15) satisfying, on the bases of the cylinder $[0, T] \times G$, the conditions

$$u(T, x_2, \dots, x_n) = \rho u(0, x_2, \dots, x_n),$$

$$u'_{x_1}(T, x_2, \dots, x_n) = \rho u'_{x_1}(0, x_2, \dots, x_n).$$

For equation (1) this problem corresponds to the problem of finding a solution under the conditions

$$u(T) = \rho u(0), \qquad u'_t(T) = \rho u'_t(0). \tag{16}$$

These conditions will be self-adjoint if and only if $|\rho| = 1$. The integral equation corresponding to conditions (16) has the form

$$y(t) = \frac{\lambda}{2}\int_0^T B^{1/2}(t)\left\{S(|t-\tau|) + [\rho I - S(T)]^{-1} S(T+t-\tau) + \right.$$

$$\left. + \left[\frac{1}{\rho} I - S(T)\right]^{-1} S(T-t+\tau)\right\} A^{-1/2} B^{1/2}(\tau) y(\tau)\, d\tau. \qquad (17)$$

Here it is assumed that ρ and $1/\rho$ are not eigenvalues of the operator $S(T) = e^{-A^{1/2}T}$. This will certainly be the case under self-adjoint conditions.

Let us formulate the corollaries that follow from the integral equation for problem (1)–(16).

1°. To every complex number ρ there corresponds a countable number of values λ for which solutions of problem (1)–(16) exist, and moreover $\lambda_n \to \infty$ as $n \to \infty$.

2°. **Analogue of the Poincaré–Lyapunov theorem.** If, for real λ, problem (1)–(16) has solutions for some ρ, then for the same λ it also has solutions for $\rho_1 = 1/\bar{\rho}$.

This assertion follows from the fact that, when ρ is replaced by $1/\bar{\rho}$, the kernel $\mathfrak{K}(t,\tau)$ of integral equation (17) goes over into the kernel of the adjoint equation $\mathfrak{K}^*(\tau,t)$. By Fredholm' s theorem, to each characteristic number λ of equation (17) there corresponds a characteristic number $\bar{\lambda}$ of the adjoint equation. Thus, if there is a solution of problem (1)–(16) corresponding to the pair of numbers (ρ,λ), then there is a solution corresponding to the pair $(1/\bar{\rho},\bar{\lambda})$. It is clear from the proof that the root subspaces corresponding to the pairs $(1/\rho,\lambda)$ and $(1/\bar{\rho},\bar{\lambda})$ have the same structure.

3°. If $|\rho| = 1$, then all eigenvalues of problem (1)–(16) are real and positive: $\lambda_n = \omega_n^2$.

4°. For $|\rho| = 1$, the following inequalities hold for the eigenvalues ω_n^2:

$$\lambda_n^{(2)} \le \omega_n^2 \le \lambda_n^{(1)},$$

where $\lambda_n^{(i)}$ are the eigenvalues of the first ($i = 1$) and second ($i = 2$) boundary-value problems.

We note that, for the waveguide equation, properties 3°–4° are known ([1]), while property 2° was formulated in ([1]) as a hypothesis.

Up to now we have assumed that the operator $B(t)$ is bounded. The presence in all the integral equations of the factor $A^{-1/2}$, as well as the known estimate $\|A^{1/2}e^{-A^{1/2}t}\| \le C/t$ ([2]), make it possible to carry out all the considerations also in the case when $B(t)$ is an unbounded positive definite operator of fractional order with respect to the operator A ([3]).

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Note: Figure translations are in progress. See original paper for figures.