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A GENERALIZATION OF KLOOSTERMAN’ S FORMULA

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Abstract

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MATHEMATICS

A. I. VINOGRADOV

A GENERALIZATION OF KLOOSTERMAN’ S FORMULA

(Presented by Academician I. M. Vinogradov on 28 IV 1962)

In 1926 Kloosterman supplemented the Hardy-Littlewood circle method with his considerations from the theory of theta-series and proved the solvability of the equation

$$n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2, \tag{1}$$

where n is an integer; a_1, a_2, a_3, a_4 are fixed integers, and

$$\left(n, \prod_1^4 a_i\right) = 1.$$

Yu. V. Linnik in (^{1}) gave a conditional solution of the generalized Kloosterman equation:

$$n = N(\mathfrak{a}) + N(\mathfrak{b}), \tag{2}$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are integral ideals of prescribed classes of certain algebraic number fields K and K_1, with one of these fields, say K_1, being quadratic, while the other K is of arbitrary degree n.

The conditional nature of Yu. V. Linnik’ s proof consisted in the fact that he made essential use of Siegel’ s unproved theorem on a special zero for L-series of the form:

$$L_K(s, \chi_1\chi_2) = \sum_{\mathfrak{a}} \frac{\chi_1(N(\mathfrak{a}))\chi_2(\mathfrak{a})}{N(\mathfrak{a})^s}, \tag{3}$$

where χ_1 is a real character of the group of norm residues modulo q (q is a rational integer), χ_2 is a real character of the character group constructed on the ideal class group of the field K. Moreover, in the asymptotic formula for the number of solutions of equation (2), given by Yu. V. Linnik in (^{1}):

$$Q(n) = \sigma(n, d, d_2)n + O\left(\frac{n}{\ln n} \ln \ln n\right), \tag{4}$$

it was not a priori clear whether the singular series $\sigma(n, d, d_2)$ is different from zero or not (d and d_2 are the discriminants of the fields K and K_1, respectively).

In this note we briefly outline proofs of Siegel' s theorem for the L-series (3) and of the theorem on the number of solutions of the equation

$$n = N(\mathfrak{p}_1\mathfrak{p}_2) + \varphi(x, y), \tag{5}$$

where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals of the field K, and, moreover, their product belongs to a prescribed class C,

$$\varphi(x, y) = ax^2 + by^2, \qquad (n, ab) = 1, \qquad (a, b) = 1.$$

Theorem 1. *If the L-series (3) has real zeros, then the largest real zero satisfies the inequality*

$$1 - \gamma > \frac{c(\varepsilon)}{|d^2q^n|^{\varepsilon}}.$$

where $\varepsilon > 0$ is an arbitrary absolute constant; $c(\varepsilon) > 0$ is an absolute constant depending only on ε.

The proof is carried out according to the following scheme: consider two products

$$\begin{gathered} \zeta_K(s)L_K(s, \chi_1\chi_2) = \zeta_{K^*}(s), \\ \zeta_K(s)L_K(s, \chi_1) = \zeta_{K_1^*}(s). \end{gathered} \tag{6}$$

These two equalities correspond to two quadratic extensions of the field K:

$$\begin{tikzcd} & K^* \\ K \arrow[ur] \arrow[dr] & \\ & K_1^*, \end{tikzcd}$$

and it follows from class-field theory that the discriminants of the fields K^* and K_1^* are the same. But the discriminant of the field K_1^* does not exceed, in absolute value, the number $|d^2qn|$.

Consequently, on the basis of the first equality (6), similarly to how this was done in paper (^{2}), we obtain Siegel' s theorem for the L-series (3).

Theorem 2. If we denote by $P(n)$ the number of solutions of equation (5) under the condition that

$$\frac{1}{2}n^{\theta} \leqslant N(\mathfrak{p}_1) \leqslant n^{\theta}, \qquad \frac{1}{2}n^{1-\theta} \leqslant N(\mathfrak{p}_2) \leqslant n^{1-\theta}, \qquad 0 < \theta \leqslant 0.01,$$

then

$$P(n) = \frac{c_0}{H\sqrt{ab}} \frac{n}{\ln^2 n} \sigma_1(n,d,a,b) + O\left(\frac{n}{\ln^3 n} \ln\ln n\right),$$

where $\sigma_1(n,d,a,b)$ is an absolutely convergent series;

$$\sigma_1(n,d,a,b) = \sum_{q=1}^{\infty} \frac{\omega(q,n,a,b)}{q_1\varphi(q_1)} \frac{(-1)^{\left(\frac{\delta-1}{2}\right)^2}\left(\frac{ab}{\delta}\right)}{\delta\varphi_K(\delta)} \sum_{(l,\delta)=1} \tau(l,\delta) e^{-2\pi i \frac{l}{\delta} n}, \tag{7}$$

$$q = q_1\delta, \qquad (q_1, d) = 1;$$

$\omega(q,n,a,b)$ is a certain, rather cumbersome multiplicative function; δ runs through all numbers whose prime divisors divide d, the discriminant of K; $\varphi_K(\delta)$ is the order of the group of norm residues modulo δ;

$$\tau(l,\delta) = \sum_{(l_K,\delta)} e^{2\pi i \frac{l_{K,\delta}}{\delta} l}; \tag{8}$$

(l_K, δ) means that $l_{K,\delta}$ runs through a complete system of norm residues of the field K modulo δ.

The proof of this theorem is carried out according to the scheme of the dispersion method, set forth in detail in (^{1}), with the use of Siegel' s theorem on the exceptional zero given here, Page' s theorem on the rare distribution of Siegel zeros, and Fogels' theorem on the boundary of all zeros for Hecke L-series.

From the exact formula (6) for $\sigma_1(n,d,a,b)$, it is clearly seen why a priori one cannot say whether the exceptional series of the generalized Kloosterman formula vanishes or not. The whole matter lies in the sum (8). For an arbitrary field K there is no possibility of computing this sum, since the law of distribution of $l_{K,\delta}$ in the group of all residues modulo δ is unknown. Only a lower bound is known for

$$\varphi_K(\delta) \geqslant \frac{\varphi(\delta)}{n^{\omega(\delta)}},$$

where $\omega(\delta)$ is the number of prime factors of δ, and n is the degree of the field. In the case of equation (2), the sum (8) also enters into a special case of formula (4) of Yu. V. Linnik. This also explains the impossibility of an a priori assertion that equations (2) and (5) can be solved for all sufficiently large n over any field

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K. Therefore, for equation (5) as well one can only assert that, for a given field K with discriminant d and group of norm residues $(l_{K,\delta})$, we can determine in a finite number of operations whether the special series of solutions of the equation tends to zero or not.

It may be noted that there are exceptions to this rule. If the discriminant of the field K is a prime number or a power of a prime, then the investigation of the special series can be carried through to the end and it can be shown that it exceeds the quantity

$$\frac{c_0'}{\ln\ln n},$$

where c_0' is an absolute positive constant depending only on the degree of the field K and on the discriminant d.

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Note: Figure translations are in progress. See original paper for figures.

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