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Abstract

Full Text

MATHEMATICS

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ON A THEOREM OF S. N. BERNSTEIN

(Presented by Academician S. N. Bernstein on 21 IV 1962)

In the paper ⁽¹⁾ the following is proved

Theorem (S. N. Bernstein). *If $f(z)$ is a real polynomial having only real zeros, and if $S(\theta)$ is a trigonometric polynomial of degree n with arbitrary complex coefficients, then for every real θ the inequality holds*

$$\left| f\left(\frac{d}{d\theta}\right) S(\theta) \right| \leq \max_{0 \leq \theta < 2\pi} |S(\theta)| \cdot |f(in)|. \quad (1)$$

In the same work S. N. Bernstein showed, by way of an example, that inequality (1) also holds for the one-parameter family of polynomials

$$f_\alpha(z) = z^2 \cos^2 \frac{\alpha}{2} + (2n-1)z \cos \frac{\alpha}{2} \sin \frac{\alpha}{2} + n(n-1) \sin^2 \frac{\alpha}{2} + \frac{n}{2} \quad (2)$$

(α is a real parameter), which have nonreal zeros. It is easy to see that, if α runs over the real axis, the zeros of the polynomials (2) describe the hyperbola

$$y^2 - \frac{x^2}{2n-1} = \frac{n}{2}. \quad (3)$$

Let H denote the domain bounded by both branches of this hyperbola. In the present note we shall establish that the cited theorem of S. N. Bernstein remains valid under a more general assumption—when the zeros of the real polynomial $f(z)$ lie in H .

In order to bring out more clearly the idea of the proof, we shall first carry out the reasoning in the case when $f(z) = z$, i.e., we shall present a new proof of S. N. Bernstein's inequality:

$$\max_{0 \leq \theta < \pi} |S'(\theta)| \leq n \max_{0 \leq \theta < 2\pi} |S(\theta)|. \quad (4)$$

The proof that we have in mind is based on the following well-known theorem of Laguerre.

If $R(z)$ is a polynomial of degree n ; ζ is an arbitrary point of the complex plane and K is any circle passing through the points ζ and ζ' , where

$$\zeta' = \zeta - \frac{nR(\zeta)}{R'(\zeta)}, \quad (5)$$

then either inside K and outside K the polynomial $R(z)$ has at least one zero each, or all zeros of $R(z)$ lie on K .

Let $L = \max |S(\theta)|$, $\omega = \cos \theta + i \sin \theta$, and $S(\theta) = \omega^{-n}Q(\omega)$, where $Q = Q(z)$ is a polynomial in z of degree $2n$. By assumption we have

$$|Q(z)| \leq L, \quad \text{when } |z| = 1. \quad (6)$$

The inequality $|S'(\theta)| \leq nL$, which we seek to prove, can be written in the new notation in the following form:

$$|nQ(z) - zQ'(z)| \leq nL, \quad |z| = 1. \quad (7)$$

Suppose that for some point ξ of the unit circle inequality (7) is false and, consequently, the equality

$$nP(\xi) - \xi P'(\xi) = 0, \quad (8)$$

holds, where $P(z) = Q(z) - \lambda$, $\lambda = Q(\xi) - \xi n^{-1}Q'(\xi)$, $|\lambda| > L$. Apply Laguerre's theorem to the polynomial $P(z)$. In view of the fact that the equality $P'(\xi) = 0$, i.e. $P'(\xi) = Q'(\xi) - \lambda = 0$, as is easy to see, leads to a contradiction, we shall consider only the case when $P'(\xi) \neq 0$. For the point ξ' , defined by (5), we obtain from (8) $\xi' = \xi - 2n\frac{\xi}{n} = -\xi$, and therefore in the disk $|z| \leq 1$ (which contains ξ and ξ') there must lie at least one zero of the polynomial $P(z)$. The latter, however, means that in the unit disk there exists a point z_0 for which $|Q(z_0)| = |\lambda| > L$, which contradicts (6) and the maximum-modulus principle. Thus inequality (4) is proved.

The case when $f(z)$ is an arbitrary real polynomial of the first degree is considered in a similar way.

Let $f(z) = az^2 + bz + c$ be any real polynomial of the second degree whose zeros are non-real and lie in H . It is not hard to verify that the latter condition is expressed by the inequality

$$nb^2 - n(2n-1)a^2 - 2ac(2n-1) \geq 0. \quad (9)$$

We shall prove that for every real θ the inequality

$$|aS''(\theta) + bS'(\theta) + cS(\theta)| \leq |f(in)|L. \quad (10)$$

holds. The last inequality, by elementary transformations, can be reduced to the following:

$$|f(-in)Q(z) + z[a(2n-1) + ib]Q'(z) - az^2Q''(z)| \leq L|f(in)|, \quad |z| = 1. \quad (11)$$

Suppose that for some ξ , $|\xi| = 1$, the last inequality is false, i.e. the equality

$$f(-in)P(\xi) + \xi[a(2n-1) + ib]P'(\xi) - a\xi^2P''(\xi) = 0, \quad (12)$$

holds, where $P(z) = Q(z) - \lambda$, $|\lambda| > L$. Further, instead of Laguerre's theorem we use the following analogous proposition:

If $R(z)$ is an arbitrary polynomial of degree n and the equality

$$AR(\xi) + BR'(\xi) + CR''(\xi) = 0, \quad (13)$$

holds, where A, B, C are any complex constants, then every circular domain containing all three points

$$\xi, \quad \xi + z_1, \quad \xi + z_2, \quad (14)$$

where z_1 and z_2 are the roots of the quadratic equation

$$Az^2 - nBz + n(n-1)C = 0, \quad (15)$$

contains at least one zero of $R(z)$.*

Apply this proposition to the polynomial $P(z)$. The quadratic equation (15) in this case will have the form

$$f(-in)z^2 - 2n\xi[a(2n-1) + ib]z - 2n(2n-1)a\xi^2 = 0, \quad (16)$$

* This proposition is not new, since in the geometry of zeros general theorems are known of which it is a consequence. It can be derived from Laguerre's theorem in the same way as Grace's theorem is usually proved, or it can be proved with the aid of this latter theorem, if one takes into account that, according to (13), the polynomials $P_1(z) = R(z + \xi)$ and $P_2(z) = Az^n - nBz^{n-1} + n(n-1)Cz^{n-2}$ are apolar.

and its roots z_1 and z_2 can be written in the following form: $z_1 = (v_1 - 1)\zeta$, $z_2 = (v_2 - 1)\zeta$, where v_1 and v_2 are the roots of the equation

$$f(-in)v^2 - (2n^2a - a + 2c)v + f(in) = 0. \quad (17)$$

A straightforward calculation shows that inequality (9) expresses, in addition, the condition that the roots of (17) lie on the unit circle; thus the points (14), i.e. the points ζ , $\zeta + z_1 = v_1\zeta$, and $\zeta + z_2 = v_2\zeta$, also lie on the circle $|z| = 1$. It follows that $P(z)$ has at least one zero in the unit disk, and thus, just as in the proof of (4), we arrive at a contradiction. This proves inequality (10).

The general case, when $f(z)$ is an arbitrary real polynomial all of whose zeros lie in H , reduces, as is known, to the successive application of the cases considered above.

Finally, we note that a careful analysis of the proposed method of proof of S. N. Bernstein's theorem makes it possible to assert that this theorem remains true also in the case when $f(z)$ is a polynomial with complex coefficients, all of whose zeros lie in the half-plane $\text{Im } z \leq 0$.

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CITED LITERATURE

1. S. N. Bernstein, A new proof of an inequality concerning trigonometric polynomials, Collected Works, **1**, article No. 30, 1952, p. 335.

Note: Figure translations are in progress. See original paper for figures.

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