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V. N. MASLENNIKOVA

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Abstract

Full Text

MATHEMATICS

V. N. MASLENNIKOVA

ON A CLASS OF SYSTEMS OF QUASILINEAR DIFFUSION EQUATIONS

(Presented by Academician S. L. Sobolev on 4 IV 1962)

We consider the first boundary-value problem for a system of quasilinear equations

$$\begin{aligned} \Lambda(u) &\equiv \sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t,u) \frac{\partial u}{\partial x_i} \\ &- \frac{\partial u}{\partial x_i} - f(x,t,u, \nabla u) = h(x,t,u,w), \\ &\frac{\partial w}{\partial t} = g(x,t,u,w), \end{aligned} \quad (1)$$

where $\nabla u = (\partial u / \partial x_1, \partial u / \partial x_2, \dots, \partial u / \partial x_n)$, with two unknown functions $u(x,t)$, $w(x,t)$ and with coefficients given in the closed cylindrical domain $\overline{D_T}$, where $D_T = \{(x,t), x \in B, 0 < t \leq T\}$; B is a bounded n -dimensional domain in the space of the variables $x = (x_1, x_2, \dots, x_n)$. The natural boundary of the domain D_T , i.e. the lateral surface S_T of the cylinder D_T and its base at $t = 0$, will be denoted by ∂D_T . It is assumed that the quasilinear operator Λ is parabolic for $(x,t) \in \overline{D_T}$, i.e. for all real vectors $(\lambda_1, \lambda_2, \dots, \lambda_n)$

$$\sum_{i,j=1}^n a_{ij}(x,t) \lambda_i \lambda_j \geq a_0 \sum_{i=1}^n \lambda_i^2.$$

System (1) is solved under the following boundary conditions:

$$u(x,t) = u_0(x,t) \quad \text{on } \partial D, \quad (2)$$

$$w(x,t) = w_0(x) \quad \text{for } t = 0. \quad (3)$$

Problems of this type arise, for example, in neutron diffusion in nuclear reactors when delayed neutrons are taken into account. For a linear operator Λ , a similar problem was studied in work (2).

The solution of the problem is sought in the classes of functions $u \in C^{2+\alpha}$, $w \in C^\alpha$, $\partial w/\partial t \in C^\alpha$, where $C^{2+\alpha}$ is the set of functions having derivatives with respect to x_k up to order 2 and with respect to t up to order 1 inclusive, satisfying a Hölder condition (with exponent α , $0 < \alpha < 1$); here the distance between two points $\bar{P}(x, t)$, $\bar{P}(\bar{x}, \bar{t})$ in the Hölder condition is taken in the form

$$d(\bar{P}, \bar{P}) = \left[\sum_{i=1}^n (\bar{x}_i - \bar{x}_i)^2 + |\bar{t} - \bar{t}| \right]^{1/2}; \quad (4)$$

C^α is the set of functions satisfying the Hölder condition with exponent α and with the same distance (4) (see (3-5)).

A. Everywhere in what follows it is assumed that $h(x, t, u, w)$ is a nondecreasing function of w , and $g(x, t, u, w)$ is a nonincreasing function of u ; the functions h, g with respect to u and w , and b_i and f with respect to u , satisfy the uniform Lipschitz condition

$$|h(x, t, u_1, w_1) - h(x, t, u_2, w_2)| \leq M(|u_1 - u_2| + |w_1 - w_2|)$$

for $(x, t) \in \bar{D}_T$ and for all values of u, w from some bounded domain.

Following paper (2), comparison theorems are proved, which will subsequently be used to prove the uniqueness and existence of the solution of problem (1)–(3).

Theorem 1. Let $\{u_1, w_1\}, \{u_2, w_2\}$ be two pairs of continuous functions defined in \bar{D}_T . Suppose that their second derivatives with respect to x_i and first derivatives with respect to t exist, are uniformly bounded in \bar{D}_T , and satisfy in \bar{D}_T the differential inequalities

$$\Lambda(u_1) - h(x, t, u_1, w_1) > \Lambda(u_2) - h(x, t, u_2, w_2),$$

$$\frac{\partial w_1}{\partial t} - g(x, t, u_1, w_1) < \frac{\partial w_2}{\partial t} - g(x, t, u_2, w_2),$$

and, moreover, $u_1 < u_2$ on ∂D_T , $w_1 < w_2$ for $t = 0$.

Then $u_1 < u_2$, $w_1 < w_2$ in \bar{D}_T .

In the proof of Theorem 1 the maximum principle is used.

Theorem 2. Let $\{u_1, w_1\}, \{u_2, w_2\}$ be two pairs of continuous functions defined in \bar{D}_T . Suppose that their second derivatives with respect to x_i and first derivatives with respect to t exist, are uniformly bounded in \bar{D}_T , and satisfy in \bar{D}_T the differential inequalities

$$\Lambda(u_1) - h(x, t, u_1, w_1) \geq \Lambda(u_2) - h(x, t, u_2, w_2),$$

$$\frac{\partial w_1}{\partial t} - g(x, t, u_1, w_1) \leq \frac{\partial w_2}{\partial t} - g(x, t, u_2, w_2).$$

Moreover, let $u_1 \leq u_2$ on ∂D_T ; $w_1 \leq w_2$ for $t = 0$.

Then $u_1 \leq u_2$, $w_1 \leq w_2$ in \overline{D}_T .

In the proof, the method of continuation with respect to a parameter and Theorem 1 are used.

With the aid of Theorem 2, the uniqueness of the solution of problem (1)–(3) is established.

Theorem 3. Problem (1)–(3) can have no more than one solution in the class of functions having bounded derivatives entering the equation.

Indeed, let $\{u_1, w_1\}, \{u_2, w_2\}$ be two such solutions satisfying the same boundary conditions. Then, by virtue of Theorem 2, we have in \overline{D}_T :

$$u_1 \leq u_2 \leq u_1, \quad w_1 \leq w_2 \leq w_1,$$

whence $u_1 \equiv u_2$, $w_1 \equiv w_2$.

Let the following assumptions on the coefficients of equation (1) and on the lateral surface of the domain \overline{D}_T be fulfilled.

I. The lateral surface S_T of the domain \overline{D}_T belongs to the classes C^2 and $C^{2+\alpha}$ (see (3)).

II. The coefficients of the operator Λ satisfy the conditions of the existence and uniqueness theorems of paper (4), i.e. $a_{ij}(x, t)$, $b_i(x, t, u)$, $f(x, t, 0, 0)$, $\partial f(x, t, u, 0)/\partial u$, $\partial f(x, t, u, p)/\partial p_i$, where $p_i = \partial u/\partial x_i$ ($i = 1, 2, \dots, n$), satisfy the Hölder condition with exponent α ($0 < \alpha < 1$) with respect to x, t ; the Lipschitz condition with respect to u and the Hölder condition with exponent β ($0 < \beta \leq 1$) with respect to p_i , and, moreover,

$$|\partial f(x, t, u, p)/\partial p_i| < c_0$$

and

$$\partial f(x, t, u, 0)/\partial u \geq b_0,$$

where c_0, b_0 are constants.

The functions $a_{ij}(x, t)$ on the lateral surface satisfy the Lipschitz condition with the ordinary distance

$$\rho(\overline{P}, \overline{\overline{P}}) = \left[\sum_{i=1}^n (\overline{x}_i - \overline{\overline{x}}_i)^2 + (\overline{t} - \overline{\overline{t}})^2 \right]^{1/2}.$$

III. The functions $h(x, t, u, w)$, $g(x, t, u, w)$, in addition to condition A, also satisfy the Hölder condition with exponent α ($0 < \alpha < 1$) in $(x, t) \in \bar{D}_T$ for any bounded values of u and w .

Theorem 4. Suppose that conditions A, I–III are fulfilled and that there exists a function $\psi(x, t) \in C^{2+\alpha}$ in \bar{D}_T , coinciding with the prescribed boundary values $u_0(x, t)$ of the function $u(x, t)$ on ∂D , while $w_0(x, 0)$ satisfies the Hölder condition with exponent α on \bar{B} ($0 < \alpha < 1$).

Suppose, moreover, that there exist pairs of functions (u', w') , (u'', w'') , Hölder continuous with exponent α and possessing continuous and bounded derivatives, satisfying the system of inequalities:

$$\begin{aligned} \Lambda(u') - h(x, t, u', w') &\geq 0 \geq \Lambda(u'') - f_2(x, t, u'', w''), \\ \frac{\partial w'}{\partial t} - g(x, t, u', w') &\leq 0 \leq \frac{\partial w''}{\partial t} - g(x, t, u'', w''), \\ u'(x, t) &\leq u_0(x, t) \leq u''(x, t) \quad \text{on } \partial D, \\ w'(x, 0) &\leq w_0(x, 0) \leq w''(x, 0) \quad \text{for } t = 0. \end{aligned} \quad (5)$$

Then there exists a solution of system (1) with boundary conditions (2), (3). Moreover, $u(x, t)$ belongs to the class $C^{2+\gamma}$ in \bar{D}_T for $\gamma \leq \alpha\beta < 1$, and w , $\partial w/\partial t$ belong to the class C^α in \bar{D}_T .

The proof of this theorem can be carried out by the method of successive approximations:

$$\begin{aligned} \Lambda(u_n) - Mu_n &= h(x, t, u_{n-1}, w_{n-1}) - Mu_{n-1}, \\ \frac{\partial w_n}{\partial t} + Mw_n &= g(x, t, u_{n-1}, w_{n-1}) + Mw_{n-1}, \\ u_n|_{\partial D} &= u_0(x, t); \quad w_n|_{t=0} = w_0(x); \quad n = 1, 2, 3, \dots \end{aligned} \quad (6)$$

As the zeroth approximation one takes the functions $u''(x, t)$, $w''(x, t)$, indicated in the hypothesis of the theorem. A system of type (6) thus splits into two separate equations. The first equation is solvable on the basis of the results of papers (^{4, 5}), and the second equation is integrated by quadratures.

With the help of the second comparison theorem, the convergence of the method of successive approximations is proved. Namely, it is proved that for all n

$$u' \leq u_n \leq u_{n-1} \leq u''; \quad w' \leq w_n \leq w_{n-1} \leq w'',$$

i.e. the sets $\{u_n\}$, $\{w_n\}$ form a monotonically decreasing sequence, bounded at each point.

It is then proved that the limiting functions u and w satisfy the Hölder condition with a Hölder constant independent of n , and that the first derivatives with

respect to x_i of the functions u satisfy the Hölder condition also with a Hölder constant independent of n . This is proved with the aid of an estimate for the solution of a quasilinear parabolic equation given in paper (5) (δ is arbitrary, $0 < \delta < 1$):

$$|u_n|_{1+\delta}^{D_T} \leq M_2 \left(\sup_{D_T} |f(x, t, 0, 0)| + |\psi|_2 + M|u_{n-1}|_0 + |h(x, t, u_{n-1}, w_{n-1})| \right),$$

where M_2 depends only on the Hölder constants in \bar{D}_T and the Lipschitz constants on S_T , the coefficients $a_{ij}(x, t)$, the constants a_0, b_0, c_0 , and the maxima of the moduli of the remaining coefficients of the operator Λ . Since the uniform boundedness of u_n, w_n implies the uniform boundedness of $h(x, t, u_{n-1}, w_{n-1})$, it follows that $|u_n|_{1+\delta}$ is bounded...

bounded above by a constant independent of n . It is then proved that the limit functions satisfy system (1). For this purpose one considers the solution u^*, w^* of the system

$$\begin{aligned} & \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u^*}{\partial x_i \partial x_j} - \frac{\partial u^*}{\partial t} - Mu^* = \\ & = \sum_{i=1}^n b_i(x, t, u) \frac{\partial u}{\partial x_i} - f(x, t, u, \nabla u) + h(x, t, u, w) - Mu, \\ & \frac{\partial w^*}{\partial t} + Mw^* = g(x, t, u, w) + Mw, \end{aligned} \tag{7}$$

$$u^*|_{\partial D} = u_0(x, t), \quad w^*|_{t=0} = w_0(x).$$

The right-hand sides of this system satisfy the Hölder condition; therefore this system has a solution

$$u^* \in C^{2+\gamma}, \quad w^* \in C^\alpha, \quad \frac{\partial w^*}{\partial t} \in C^\alpha$$

(see (3, 5)).

In fact, $u^* = u$.

Indeed, since the set $\{u_n\}$ is compact in $C^{1+\delta_1}$ ($0 < \delta_1 < \delta < 1$), we choose from it a subsequence converging in this metric. Writing system (6) for the n_i -th term of this subsequence, subtracting it from system (7), and using estimates for the linear parabolic equation and the convergence of $\{u_{n_i}\}$ in the metric $C^{1+\delta_1}$, we obtain that $u^* = u$. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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