MATHEMATICS

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1962

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Abstract

Full Text

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ON CUBATURE FORMULAS ON THE SPHERE INVARIANT UNDER TRANSFORMATIONS OF FINITE GROUPS OF ROTATIONS

A cubature formula on the surface of the sphere

$$(l, f) = \iint_S f(\vartheta, \varphi)\, dS - \sum C_k f(x^{(k)}) \cong 0 \tag{1}$$

is called invariant with respect to transformations of rotations of the sphere from some group G, if

$$(l, f(\vartheta_1(\vartheta, \varphi), \varphi_1(\vartheta, \varphi))) = (l, f(\vartheta, \varphi)), \tag{2}$$

where

$$\vartheta_1(\vartheta, \varphi),\ \ \varphi_1(\vartheta, \varphi) \tag{3}$$

is a substitution from G.

L. A. Lyusternik and V. A. Ditkin ([1,2]) considered formulas with vertices at the vertices and centers of the faces of the icosahedron. We shall indicate how to construct invariant formulas that are valid for the largest possible number of spherical harmonics ([3]) for the groups of rotations of the sphere corresponding to regular polyhedra.

Theorem 1. *In order that a cubature formula invariant with respect to the group G be exact for all harmonics of a given order, it is necessary and sufficient that it be exact for all invariant harmonics $Y_n^*(\vartheta, \varphi)$, i.e., for those which do not change under all rotations of the sphere belonging to the group G:*

$$Y_n^*(\vartheta_1(\vartheta, \varphi), \varphi_1(\vartheta, \varphi)) \equiv Y_n^*(\vartheta, \varphi). \tag{4}$$

The proof is based on the formula

$$(l, f) = (l, f_G), \tag{5}$$

where f_G is the arithmetic mean of the function f over the group,

$$f_G = \frac{1}{M} \sum_{g \in G} f(gx). \tag{6}$$

Let us denote the number of such invariant harmonics by $S(n)$. This number, as D. K. Faddeev pointed out to the author, can be calculated using representation theory.

Spherical harmonics of order n form a $(2n+1)$-dimensional space, for which one may take as a basis, for example,

$$e^{im\varphi} P_n^{(m)}(\cos\vartheta) \quad (m = -n, -n+1, \dots, 0, 1, \dots, n). \tag{7}$$

The group of rotations of the sphere realizes a group of linear substitutions A of these elements, which is a linear representation of this group. Every representation decomposes into irreducible representations in subspaces of lower dimension. Among them there is a certain number of identity representations. The number $S(n)$ of linearly independent invariant har-

coincides with the number of such identical one-dimensional representations contained in the representation A.

The traces of the matrices of irreducible representations, the so-called characters of the representation, form an M-dimensional vector. As is known, the characters of distinct irreducible representations are orthogonal:

$$\sum_{k=1}^{M} \chi\left(A_k^{(j)}\right) \overline{\chi\left(A_k^{(s)}\right)} = \begin{cases} M, & A^{(j)} \sim A^{(s)}, \\ 0, & A^{(j)} \nsim A^{(s)}. \end{cases} \tag{8}$$

All characters of the identity representation will obviously be equal to one.

Hence for the number $S(n)$ we obtain the formula

$$S(n) = \frac{1}{M} \sum_{k=1}^{M} \chi(A_k), \tag{9}$$

where A_k are the matrices representing the elements of the group.

The traces of similar matrices coincide, and in any rotation group of the sphere, rotations through equal angles about analogous elements are similar. In every regular polyhedron there are t_1 vertices, t_2 faces, and t_3 edges. At the vertices q_1 elements meet; the faces are regular q_2-gons; and the edges serve as axes of rotations of order $q_3 = 2$ of the corresponding group G.

The following relations are evident:

$$t_1 q_1 = t_2 q_2 = t_3 q_3 = M, \tag{10}$$

and also

$$\frac{1}{2}\left[t_1(q_1-1)+t_2(q_2-1)+t_3(q_3-1)\right]+1 = M,$$

whence

$$t_1 + t_2 + t_3 = M + 2. \tag{11}$$

The sums of the traces of all rotations, including the identity, about some vertex, the center of a face, and the midpoint of an edge will be equal to:

$$\sum_{k=0}^{q_j} \sum_{r=-n}^{+n} e^{2k\pi i/q_j} = q_j \left(2\left[\frac{n}{q_j}\right]+1\right). \tag{12}$$

Adding them over all axes and observing that in doing so the identity rotation is counted $M+2$ times and all the others twice, using (9) and (10), we obtain the theorem:

Theorem 2.

$$S(n) = \left[\frac{n}{q_1}\right] + \left[\frac{n}{q_2}\right] + \left[\frac{n}{q_3}\right] - n + 1. \tag{13}$$

A simple calculation leads to the corollary:

$$S\left(\frac{M}{2}-n-1\right) + S(n) = 1, \qquad 0 \le n \le \frac{M}{2} - 1. \tag{14}$$

We transform formula (13) into another form. Let Q^* be the set of those q_i for which $n \not\equiv 0 \pmod{q_i}$.

Expressing the integer part through the fractional part, we give $S(n)$ the form

$$S(n) = 1 + \frac{2n - \sum\limits_{q_j \in Q^*} t_j}{M} - \sum_{q_j \in Q^*} \left(\left\{\frac{n}{q_j}\right\} - \frac{1}{q_j}\right). \tag{15}$$

Since

$$\sum_{q_j \in Q^*} \left(\left\{ \frac{n}{q_j} \right\} - \frac{1}{q_j} \right) < 1,$$

it follows from (15) that

$$S(n) = \left[1 + \frac{2n - \sum\limits_{q_j \in Q^*} t_j}{M} \right], \tag{16}$$

and, as a consequence,

$$S(n) = \begin{cases} \left[\dfrac{2n+1}{M} \right], & 2n+1 \leqslant \sum\limits_{q_j \in Q^*} t_j, \\ \left[\dfrac{2n+1}{M} \right] + 1, & 2n+1 > \sum\limits_{q_j \in Q^*} t_j. \end{cases} \tag{17}$$

From (17) we obtain:

$$S\left(n + \frac{M}{2}\right) = S(n) + 1. \tag{18}$$

Formula (17) admits a simple interpretation, based on the following theorem.

Theorem 3. Let X_k, Y_k, Z_k, $k = 1, 2, \dots, 2n+1$, be orthogonal coordinate systems with a common origin, and let all directions of the axes Z_k be distinct.

Then the system of functions

$$\xi_k^n = (X_k + iY_k)^n \tag{19}$$

forms a basis in the space of spherical functions of order n.

Introduce the complex variable $z = X + iY$, which maps the sphere onto the plane by means of stereographic projection. For ξ_k^n one obtains the formula:

$$\xi_k = \frac{2z_k}{1 + |z_k|^2}, \tag{20}$$

where

$$z_k = \frac{a_k z - \bar{c}_k}{c_k z + \bar{a}_k}, \qquad |a_k|^2 + |c_k|^2 = 1. \tag{21}$$

The fractional-linear substitution (21) represents the expression for a rotation of the sphere that takes the axis Z_k to the origin of coordinates.

From (20), by means of simple calculations, one obtains the expression:

$$\xi_k^n = \sum_{m=-n}^{+n} a_k^{n-m} \bar{c}_k^{n+m} R_n^{(m)}(z, \bar{z}), \tag{22}$$

where the functions

$$R_n^{(m)}(z, \bar{z}) = c_n^{(m)} e^{im\varphi} P_n^{(m)}(\cos\vartheta) \tag{23}$$

differ only by a constant factor from the basis elements (7). Thus, in our basis the function ξ_k^n corresponds to the vector

$$\xi_k^n \sim (a_k^{2n}, a_k^{2n-1}\bar{c}_k, \dots, \bar{c}_k^{2n}). \tag{24}$$

From (24), by virtue of the expression for the Vandermonde determinant, Theorem 3 follows.

For any $(2n+2)$ harmonics of the form ξ_k^n with distinct directions Z_k, from (24) we shall have:

$$\sum_{k=1}^{2n+2} \frac{\xi_k^n}{\prod_{j\neq k}(a_k\bar{c}_j - a_j\bar{c}_k)} = 0. \tag{25}$$

Consider the set

$$\{g_\alpha x^{(k)}\}, \tag{26}$$

where

$$x^{(1)},\ x^{(2)}, \dots,\ x^{\left[\frac{2n+1}{M}\right]} \tag{27}$$

a system of mutually inequivalent points in general position. Together with some system independent of them,

$$x^{(2n+1)},\ x^{(2n)},\ \dots,\ x^{(2n+\nu-M\left[\frac{2n+1}{M}\right])} \tag{28}$$

they form a system of $(2n+1)$ independent directions of the axis Z_k, and the corresponding functions

$$\xi_k^n = (X_k + iY_k)^n$$

form a basis. It is easy to see that the functions

$$\frac{1}{M} \sum_{g_\alpha \in G} \left(\zeta\left(g_\alpha x^{(k)}\right)\right)^n, \qquad k = 1, 2, \ldots, \left[\frac{2n+1}{M}\right] \tag{29}$$

(the notation is self-evident) are linearly independent invariant harmonics of order n. If, as (28), one can choose all those directions of axes whose rotations in the group correspond to $q_j \in Q^*$, which is possible when

$$(2n+1) \leq \sum_{q_j \in Q^*} t_j,$$

then it is clear that (29) exhausts all invariant spherical harmonics of order n, since the group average of each of (28) is zero. We obtain a direct proof of the first half of formula (7). If this is impossible, i.e.

$$(2n+1) > \sum_{q_j \in Q^*} t_j,$$

then for (28) one may take some system of mutually equivalent points, and the group average of any of the corresponding ξ_k^n gives one more invariant harmonic of order n.

Let us also consider the groups G^* of rotations together with reflections. For such groups the following holds.

Theorem 4. *There exist no invariant harmonics of odd order for the groups G^*. The invariant harmonics of even order for G^* coincide with the invariant harmonics for G.*

In conclusion we give a table of the values $S(n)$ for the groups $G_{\mathrm{IV}}, G_{\mathrm{VIII}}$, and G_{XX} of rotations of the tetrahedron, octahedron, and icosahedron for the values $0 \leq n < M/4$.

n	S_{IV}	S_{VIII}	S_{XX}	n	S_{XX}	n	S_{XX}
0	1	1	1	6	1	11	0
1	0	0	0	7	0	12	1
2	0	0	0	8	0	13	0
3		0	0	9	0	14	0
4		1	0	10	1		
5		0	0				

Received
29 V 1962

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2. L. A. Lyusternik, DAN, 62, No. 4, 449 (1949).
3. S. L. Sobolev, DAN, 146, No. 1 (1962).

Note: Figure translations are in progress. See original paper for figures.