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Abstract

Full Text

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ON INTEGRAL OPERATORS IN ORLICZ SPACES

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Let G be a bounded closed set in n-dimensional Euclidean space, and let G_1 be such a set in m-dimensional Euclidean space. We shall denote points of these spaces by s and t, respectively. By $\widehat{G}$ we shall denote the topological product $G_1 \times G$.

Let $P(u)$ and $Q(u)$ $(-\infty < u < \infty)$ be even, continuous functions, monotonically increasing for $u \geqslant 0$, with

$$\lim_{u\to\infty} P(u) = \lim_{u\to\infty} Q(u) = \infty,$$

$$P(0) = Q(0) = 0$$

(the last condition is inessential).

Consider the integral operator

$$Au(t) = \int_G K(t,s)u(s)\,ds, \tag{1}$$

whose kernel $K(t,s)$ is defined on $\widehat{G}$ and satisfies the conditions

$$\int_G P[K(t,s)]\,ds \leqslant C_1, \qquad \int_{G_1} Q[K(t,s)]\,dt \leqslant C_2. \tag{2}$$

Criteria are established for the continuity and complete continuity of the operator (1) as an operator from the given Orlicz space $L_M^*(G)$ into some Orlicz space $L_{M_1}^*(G_1)$ (for all definitions and statements from the general theory of Orlicz spaces used here, see (^{1})).

For the case of the spaces L_p, criteria of this type were obtained by L. V. Kantorovich (^{2}), and then supplemented by H. L. Smolitskii (^{3}).

1. Let $\Phi_1(u)$ and $\Phi_2(u)$ $(-\infty < u < \infty)$ be two functions of the same type as the functions $P(u)$ and $Q(u)$ defined above. We shall say that $\Phi_1(u)$ is “less than” $\Phi_2(u)$ (and write $\Phi_1(u) \prec \Phi_2(u)$) if, for large values of u, the inequality $\Phi_1(ku) \leqslant \Phi_2(u)$ holds, where k is some positive number. We

shall say that $\Phi_1(u)$ is "essentially less than" $\Phi_2(u)$ (and write $\Phi_1(u) \prec\prec \Phi_2(u)$) if

$$\lim_{u\to\infty} \frac{\Phi_2(\varepsilon u)}{\Phi_1(u)} = \infty$$

for every $\varepsilon > 0$.

The functions $\Phi_1(u)$ and $\Phi_2(u)$ will be called equivalent (and we write $\Phi_1(u) \sim \Phi_2(u)$) if simultaneously $\Phi_1(u) \prec \Phi_2(u)$ and $\Phi_2(u) \prec \Phi_1(u)$.

If a function $\Phi(u)$ of the type under consideration is such that $\Phi(u)/u$ tends monotonically to infinity for large u, then it is equivalent to some N-function.

Lemma. *In order that an N-function $N_1(v)$, complementary to the N-function $M_1(u)$, satisfy the Δ_2-condition, it is necessary and sufficient that there exist an N-function $M_2(u)$ such that*

$$M_1(kuv) \geqslant M_1(u)M_2(v) \tag{3}$$

for $u, v \geqslant u_0 > 0$ and some $k > 0$.

This lemma is a consequence of certain assertions of T. Ando (^{4}). However, it is also proved quite simply directly.

2. From Hölder' s inequality for Orlicz spaces there follows

Theorem 1. *Let the N-function $N(v)$, complementary to the N-function $M(u)$, satisfy the relation*

$$N(v) \prec P(v).$$

Then the operator (1) is a continuous operator acting from the space $L^*_M(G)$ into the space $M(G_1)$ of functions bounded on G_1.

It is just as simply proved that:

Theorem 2. *Let $Q(u)$ be equivalent to some N-function $Q_1(u)$. Then the operator (1) is a continuous operator acting from $L_1(G)$ into $L^*_{Q_1}(G_1)$.*

If $Q(u) \sim |u|$, then the operator (1) is a continuous operator acting from $L_1(G)$ into $L_1(G_1)$.

Below we shall assume that from the condition $\Phi_1(u) \prec\prec \Phi_2(u)$ it follows that the functions $\Phi_2(u)/\Phi_1(u)$ and $\Phi_1^{-1}(v)/\Phi_2^{-1}(v)$, tending to infinity, are monotone for large values of u and v. Here and below $f^{-1}(v)$ $(v \geq 0)$ denotes the function inverse to the monotone function $f(v)$.

Let us now consider the case where $P(v) \prec\prec N(v)$. By the assumption made, the function

$$R^{-1}(v) = v/N^{-1}[P(v)]$$

is monotone for large v, and the function $R(u)$ is equivalent to some N-function.

Theorem 3. *Let* $P(v) \prec\prec N(v)$ *and*

$$N\left[\frac{v}{M^{-1}[Q(v)]}\right] \prec P(v). \tag{4}$$

Let the function $Q[R(u)]$ *be equivalent to some* N*-function* $M_1(u)$ *satisfying the* Δ'*-condition. Let the* N*-function* $N_1(v)$ *complementary to it satisfy the* Δ_2*-condition. Let* $M_1(u) \sim M(u)$ *or* $M_1(u) \prec\prec M(u)$.

Then the operator (1) is a continuous operator acting from $L_M^*(G)$ *into any Orlicz space* $L_{M_2}^*(G_1)$*, where the* N*-function* $M_2(u)$ *satisfies the inequality (3).*

If the N*-function* $N_1(v)$ *satisfies the* Δ'*-condition, then the operator (1) is continuous as an operator from* $L_M^*(G)$ *into* $L_{M_1}^*(G_1)$.

If the functions $P(u)$, $Q(u)$, and $M(u)$ are powers, then we obtain the theorem of L. V. Kantorovich (^{2}).

We note that in the case where $v \prec P(v)$ and $v \prec Q(v)$, condition (4) does not need to be checked, since it is fulfilled automatically.

Example 1. Let $P(u) \sim e^{|u|}$, $Q(u) \sim |u|^\alpha \ln|u|$ $(\alpha > 1/2)$, and

$$M(u) \sim |u|\sqrt{\ln|u|}.$$

Then, by Theorem 3, the operator (1) is continuous as an operator from $L_M^*(G)$ into $L_{2\alpha}(G_1)$.

Let now

$$P(v) \prec\prec N\left[\frac{v}{M^{-1}[Q(v)]}\right] \prec v. \tag{5}$$

If $Q(v) \sim |v|$, then the operator (1), as was noted above, acts continuously from $L_1(G)$ into $L_1(G_1)$.

Consider the case where $v \prec\prec Q(v)$. Since it follows from (5) that $P(v) \prec\prec Q(v)$, by the assumption made the function

$$\Phi(v) = Q(v)/P(v)$$

is monotone for large v. Put

$$\widetilde{M}_1^{-1}(v) = \Phi^{-1}(v)/P[\Phi^{-1}(v)].$$

The function $\widetilde{M}_1(u)$ is equivalent to some N-function $M_1(u)$. Moreover, $M_1(u) \prec\prec M(u)$, if $M(u)$ satisfies the Δ'-condition.

Theorem 4. Let (5) be satisfied. Let the N-function $M_1(u)$ satisfy the Δ'-condition, and let the N-function $N_1(v)$ complementary to it satisfy the Δ_2-condition.

Then the assertion of Theorem 3 holds, where $M(u) = M_1(u)$.

If $M(u)$ satisfies the Δ'-condition, then $L^*_M(G) \subset L^*_{M_1}(G)$ and operator (1) is continuous, as is the operator from $L^*_M(G)$ into $L^*_{M_2}(G)$.

The last assertion is, for the case of the spaces L_p, the content of a theorem of H. L. Smolitskii (^{3}).

Example 2. Let $P(u) \sim \ln|u|$, $Q(u) \sim |u|^3/\ln|u|$. From Theorem 4 it follows in this case that operator (1) is continuous as an operator from the space $L^*_{M_1}(G)$, where $M_1(u) = u^3 \ln u$, into the space $L_3(G_1)$.

Theorems 3 and 4 essentially cover only the cases of Orlicz spaces defined by N-functions equivalent to functions of the form

$$|u|^\alpha (\ln|u|)^{\gamma_1} (\ln\ln|u|)^{\gamma_2} \dots (\ln\ln\dots\ln|u|)^{\gamma_n} \quad (\alpha \geq 1,\ \gamma_1, \dots, \gamma_n \geq 0).$$

In the study of nonlinear integral equations with "strong" nonlinearities (for example, exponential ones), there is interest in such linear operators (1) which act in Orlicz spaces $L^*_{M_1}(G_1)$, where $M_1(u)$ is a rapidly increasing N-function, for example, equivalent to $e^{|u|}$, e^{u^2}, and so on.

We give one theorem covering these cases.

Theorem 5. Let $Q(u)$ be equivalent to some N-function $Q_1(u)$ satisfying the Δ_2-condition. Let $N(u) \prec M(u)$ and let $N(u)$ satisfy the Δ'-condition. Finally, let

$$\frac{M(u)}{M^{-1}(u)} M^{-1}\left[Q^{-1}\left(\frac{M(u)}{M^{-1}(u)}\right)\right] \prec P(u). \tag{6}$$

Then operator (1) is a continuous operator acting from $L^*_M(G)$ into $L^*_{M_1}(G_1)$, where $M_1(u) \sim Q[M(u)]$.

Remark. From Theorem 2 it follows that operator (1), under the conditions of Theorem 5, simultaneously acts from the wider space $L_1(G)$ into the space $L^*_{Q_1}(G_1)$, which is wider in comparison with $L^*_{M_1}(G_1)$.

Example 3. Let $Q(u) \sim e^{|u|}$, $P(u) \sim |u|^{3/2}\sqrt{\ln|u|}$. From Theorem 5 it follows in this case that operator (1) is continuous as an operator from $L_2(G)$ into $L^*_{M_1}(G_1)$, where $M_1(u) \sim e^{u^2}$.

In proving Theorems 3 and 4 it is convenient to use the following assertion, which is of independent interest.

Theorem 6. Let $w(t,s) \in L^*_M(\hat{G})$. Let $M_1(u) \prec M(u)$. Consider the function $\varphi(t)$—the norm of the function $w(t,s)$ as a function of s in the space $L^*_{M_1}(G)$: $\varphi(t) = \|w(t,s)\|_{M_1}$.

In order that $\varphi(t) \in L^*_{M_2}(G_1)$ for any function $w(t,s) \in L^*_M(\hat{G})$, it is necessary and sufficient that, for large u and v, the inequality

$$M(kuv) \geq M_1(u) M_2(v). \tag{7}$$

hold.

The sufficiency of the condition of this theorem for the case $M_1(u) = M_2(u) = M(u)$ was proved in (5).

3. **Theorem 7.** Let the N-function $N(v)$, complementary to $M(u)$, satisfy the Δ_2-condition. Then, under the conditions of Theorem 1, operator (1) is a completely continuous operator acting from $L^*_M(G)$ into any Orlicz space $L^*_{M_1}(G_1)$ (and into $L^*_1(G_1)$).

Theorem 8. Under the conditions of Theorem 2, operator (1) is a completely continuous operator acting from any Orlicz space $L^*_M(G)$ into $L^*_{Q_1}(G_1)$.

Other conditions for the complete continuity of operator (1) can be obtained if the statements of Theorems 3, 4, and 5 are supplemented by the following assertion, which is a generalization to Orlicz spaces of a theorem proved by M. A. Krasnosel' skii and E. I. Pustyl' nik (6).

We shall call the Orlicz space $L^*_M(G)$ **essentially narrower** than the space $L^*_{M_1}(G)$ if $M(u) \ll M_1(u)$. In this case the space $L^*_{M_1}(G)$ will be called **essentially wider** than $L^*_M(G)$.

Theorem 9. *Let operator (1) act from $L^*_M(G)$ into $L^*_{M_1}(G_1)$. Let the operator*

$$A^*_+ \psi(s) = \int_{G_1} |K(t,s)|\, |\psi(t)|\, dt$$

act from $L^*_{N_1}(G_1)$ into $\overline{E}_N(G)$. Then operator (1) is a completely continuous operator from $L^*_M(G)$ into any space $L^*_{\overline{M}1}(G_1)$ essentially wider than $L^*_{M_1}(G_1)$, and from any space $L^*_{\overline{M}}(G)$ essentially narrower than $L^*_M(G)$, into $L^*_{M_1}(G_1)$.

A particular case of this assertion is formed by the theorems of L. V. Kantorovich (2) and Kh. L. Smolitskii (3) on the complete continuity of operator (1) in L_p.

From Theorems 3–5 follow theorems on operators of potential type. These theorems, by the known method, can be applied to obtain embedding theorems in Orlicz spaces. Some embedding theorems for Orlicz spaces were obtained in (7–9) and in other works.

4. One can formulate theorems analogous to those proved here for spaces of vector-functions, and also for matrix operators acting in coordinate Orlicz spaces.

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CITED LITERATURE

1. M. A. Krasnosel' skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, 1958.
2. L. V. Kantorovich, *UMN*, **11**, no. 2 (1956).
3. Kh. L. Smolitskii, *UMN*, **12**, no. 4 (1957).
4. T. Ando, *Math. Ann.*, **140**, 174 (1960).
5. D. V. Salekhov, *Ukr. Mat. Zh.*, **13**, no. 4 (1961).
6. M. A. Krasnosel' skii, E. I. Pustyl' nik, *DAN*, **142**, no. 1 (1962).
7. E. P. Kalugina, *Convex Functionals of Many-Parameter Integration*, L., 1952.
8. I. V. Gel' man, *DAN*, **122**, no. 4 (1958).
9. Ding Shia-shi, *Sci. Rec.*, **2**, no. 2 (1958).

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