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Abstract

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ON LINEAR INTEGRALS OF THE EQUATIONS OF MOTION OF A HEAVY RIGID BODY ABOUT A FIXED POINT

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S. A. Chaplygin ([1]) came to the conclusion that "the problem under consideration admits no particular linear integral in any cases other than those known hitherto." However, in the particular solutions found recently ([2], [3]) there are linear integrals. In the present article that part is set forth of the investigation of the conditions for the existence of a linear integral in which the solutions ([2], [3]) are derived from the premises of ([1]).

1°. We adopt the formulation of the problem and the notation of ([1]). Let x, x', x'' be the projections of the principal moment of momentum onto coordinate axes rigidly attached to the body, whose origin is placed at the fixed point; $\gamma, \gamma', \gamma''$ the cosines of the angles with these axes of the direction of the force of gravity; p, q, r the components of the angular velocity of the body; l, l', l'' the coordinates of its center of gravity; for simplicity of notation we take the weight of the body to be equal to unity. The equations of motion of the body are as follows ([1]):

$$\frac{dx}{dt} = rx' - qx'' + l'\gamma'' - l''\gamma', \qquad \frac{dx'}{dt} = px'' - rx + l''\gamma - l\gamma'', \tag{1}$$

$$\frac{dx''}{dt} = qx - px' + l\gamma' - l'\gamma,$$

$$\frac{d\gamma}{dt} = r\gamma' - q\gamma'', \qquad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma, \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma'. \tag{2}$$

It is sufficient to establish the conditions under which these equations have only one linear integral, since these equations admit two linear integrals only in the Bobylev–Steklov case ([4]) and in the motion of a physical pendulum, and three in the uniform rotation of the body about a fixed axis.

By a choice of coordinate axes the assumed linear integral can always be brought to the form

$$x'' = \text{const} = n. \tag{3}$$

By rotating the coordinate axes about the axis corresponding to the component x'', it is always possible to eliminate in the expression for the kinetic energy of the body the term involving the product xx':

$$T = \frac{1}{2}\left(ax^2 + a'{x'}^2 + a''x''^2\right) + (bx + b'x')x''.$$

The required result can be obtained even under the restrictions

$$b' = 0, \qquad l' = 0, \tag{4}$$

which we adopt in the present article.

The components of the angular velocity are related to the quantities x, x', x'' by the dependences

$$p = \partial T/\partial x = ax + bx'', \qquad q = \partial T/\partial x' = a'x', \qquad r = \partial T/\partial x'' = a''x'' + bx.$$

From (3) it follows that the derivatives dx''/dt and d^2x''/dt^2 vanish:

$$(a' - a)xx' - nbx' + l\gamma' = 0, \tag{5}$$

$$\begin{gathered}[(2a - a')bx + 2nlb]\gamma'' + \{[(a' - a)l'' - bl]x - n(bl'' + a'l)\}\gamma- \\ -(a' - a)l''x\gamma' + (a' - a)({x'}^2 - x^2)bx + n(a' - a)[(a'' - a'){x'}^2+ \\ +(a - a'')x^2] + nb^2x^2 + n^2(a' + a'' - 2a)bx - n^3b^2 = 0.\end{gathered} \tag{6}$$

The three general integrals of equations (1), (2) are known:

$$\frac{1}{2}(ax^2 + a'{x'}^2) + bnx - (l\gamma + l''\gamma'') = h; \tag{7}$$

$$x\gamma + x'\gamma' + n\gamma'' = m; \tag{8}$$

$$\gamma^2 + \gamma{a'}^2 + \gamma''^2 = 1. \tag{9}$$

The problem consists in finding those values of the quantities

$$a,\ a',\ a'',\ b,\ l,\ l'',\ h,\ m,\ n, \tag{10}$$

for which equations (5)–(9) determine nonconstant values $x,\ x',\ \gamma,\ \gamma',\ \gamma''$.

2°. Substitution into (6), (9) of the dependences $\gamma,\ \gamma',\ \gamma''$ on $x,\ x'$, found from (5), (7), (8), leads to the equations

$$P_2 {x'}^2 + P_4 = 0, \qquad Q_2 {x'}^4 + Q_4 {x'}^2 + Q_6 = 0.$$

Here $P_k,\ Q_k$ are polynomials in x, whose degrees are equal to the corresponding indices. Eliminating x', we arrive at the relation

$$Q_2 P_4^2 - Q_4 P_2 P_4 + Q_6 P_2^2 = 0, \tag{11}$$

which must be an identity in x. Equating to zero the coefficients of the various powers of x, we obtain equations for determining the quantities (10). Thus, the coefficient of x^{10} gives

$$l^2 l''^2 (a' - a) \left[\frac{1}{4}(2a - a')^2 l^2 + (a' - a)^2 l''^2\right] \times$$

$$\times \{[a(a' - a) + b^2] l + 2b(a' - a) l''\} = 0.$$

We satisfy this equation by setting

$$l'' = -\frac{b^2 + a(a' - a)}{2b(a' - a)}\, l, \tag{12}$$

as a result of which the coefficient of x^9 in (11) takes the form

$$\frac{nl^2(b^2 + a^2)}{32b^5(a' - a)}[b^2 + (a' - a)^2][b^2 + a(a' - a)]^2[b^2 - a(a' - a)] \times$$

$$\times [3b^2 - (a' - a)(a - 2a'')] = 0.$$

The remaining equations, when

$$b^2 = \frac{1}{3}(a' - a)(a - 2a''), \tag{13}$$

are satisfied by the values

$$m = \frac{n^3 b}{3a' l}(3a' - a - a''), \qquad h = n^2 \left[\frac{a}{2} - \frac{(a + a'')^2}{9a'}\right],$$

$$n^4 = \frac{27{a'}^2 l^2}{(a' - a)(a - 2a'')(3a' - a - a'')^2},$$

and when

$$b^2 = a(a' - a) \tag{14}$$

the corresponding quantities are as follows:

$$m = \frac{n^3 b}{{a'}^2 l}(a' - a - a'')^2, \qquad h = \frac{n^2}{2a'}[(a + a'')^2 - a'(a + 2a'')],$$

$$n^4 = \frac{{a'}^4 l^2}{(a' - a)(a' - a - a'')^2[(a' - a)(a + a'')^2 + a{a'}^2]}.$$

3°. Let $O\xi\eta\zeta$ be the principal axes of the inertia ellipsoid of the body for the fixed point O, and let φ be the angle through which these axes must be rotated about $O\eta$ in order to coincide with the axes introduced in item 1°. Let, further, A, B, C be the moments of inertia of the body with respect to the principal axes, and let $(\xi_0, 0, \zeta_0)$ be the coordinates of the center of gravity of the body. Then

$$l = \xi_0 \cos\varphi - \zeta_0 \sin\varphi, \qquad l'' = \xi_0 \sin\varphi + \zeta_0 \cos\varphi, \tag{15}$$

$$a \cos^2\varphi + a'' \sin^2\varphi + 2b \cos\varphi \sin\varphi = \frac{1}{A}, \qquad a' = \frac{1}{B},$$

$$a \sin^2\varphi + a'' \cos^2\varphi - 2b \cos\varphi \sin\varphi = \frac{1}{C}, \tag{16}$$

$$(a'' - a) \sin 2\varphi + 2b \cos 2\varphi = 0.$$

Substitution into (12) of the values l, l'', a, a', b determined from (16), (13), (15) leads to the condition

$$\xi_0(2C - A)\sqrt{A(C - B)(2C - A)} - \zeta_0(2A - C)\sqrt{C(B - C)(2A - C)} = 0, \tag{17}$$

which characterizes solution (2). If, instead of (13), equation (14) is taken, then we obtain the condition

$$\xi_0\sqrt{B-C}-\zeta_0\sqrt{A-B}=0, \tag{18}$$

which characterizes solution (3).

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Note: Figure translations are in progress. See original paper for figures.