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Abstract

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ON THE COMPLETENESS OF SYSTEMS OF ANALYTIC FUNCTIONS COMPOSED OF SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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Let $y(\lambda, z)$ be a solution of the second-order differential equation

$$P_0(z)y'' + P_1(z)y' + P_2(z)y = \lambda^2 y, \tag{1}$$

where $P_i(z)$ $(i = 0, 1, 2)$ are certain analytic functions and λ is a parameter. Denote by $\{\lambda_n\}$ a sequence of positive numbers possessing finite density τ: $\lim_{n\to\infty} \frac{n}{\lambda_n} = \tau$. Let D be a domain in which the coefficients $P_i(z)$ $(i = 0, 1, 2)$ are regular and the coefficient $P_0(z)$ does not vanish. We consider the question of the completeness of the system of solutions $\{y(\lambda_n, z)\}$ of equation (1), when the function $y(\lambda, z)$ at some point $z_0 \in D$ satisfies the initial conditions: $y(\lambda, z_0) = \gamma_0(\lambda)$, $y'(\lambda, z_0) = \gamma_1(\lambda)$, where $\gamma_0(\lambda), \gamma_1(\lambda)$ are entire functions of exponential type, and the closely related question of the properties of sequences of linear aggregates

$$Q_n(z) = \sum_{j=1}^{q_n} a_j^{(n)} y(\lambda_j, z) \qquad (n = 1, 2, ...), \tag{2}$$

formed from these solutions.

B. Ya. Levin (^{1}) showed that if, in the equation $y'' + P(z)y = \lambda^2 y$, the coefficient $P(z)$ is an even entire function, then the system of solutions $\{y(\lambda_n, z)\}$ of this equation with initial conditions $y(\lambda, 0) = 0$, $y'(\lambda, 0) = \lambda$ is complete in the domain $0 < \mathrm{Re}(z) < \pi\tau$. A. F. Leont' ev (^{2}) considered the properties of a sequence of linear aggregates formed from the solutions $y(\lambda_n, z)$ of such an equation, under the assumption that it converges uniformly in a domain in which the system $\{y(\lambda_n, z)\}$ is not complete. We shall not assume that $P(z)$ is an entire function. In investigating the completeness of the system of solutions $\{y(\lambda_n, z)\}$ of equation (1), we pass from equation (1) to an equation of the form

$$u'' + q(t)u = \lambda^2 u \tag{3}$$

and from the solutions $y(\lambda_n, z)$ $(n = 1, 2, ...)$ to the corresponding solutions $u(\lambda_n, t)$ $(n = 1, 2, ...)$ of equation (3). Such a passage is always possible. By means of the change of variable

$$t = t(z) = \int_{z_0}^{z} \frac{dz}{\sqrt{P_0(z)}} \tag{4}$$

equation (1) is reduced to the form $w_t'' + P_1^*(t)w' + P_2^*(t)w = \lambda^2 w$, and then, by means of the substitution $u(t) = \exp\left[\frac{1}{2}\int_a^t P_1^*(\xi)\,d\xi\right] w(t)$ (here a is a constant), it is reduced to the form (3).

Choose some single-valued branch of the function $t(z)$, making the necessary cuts in the z-plane. Denote by D_0 the domain, co-

which is obtained from the domain D by removing the cuts contained in it. We shall carry the further consideration over from the domain D to the domain D_0. The transformation (4) maps the domain D_0 onto the domain T, and the point $z_0 \in D_0$ onto the point $t_0 = 0$.

The solution $u(\lambda, t)$ of equation (3) corresponding to the solution $u(\lambda, z)$ of equation (1) satisfies at the point t_0 the initial conditions $u(\lambda; t_0) = \gamma_0^*(\lambda)$; $u'(\lambda, t_0) = \gamma_1^*(\lambda)$; moreover, as is not hard to show, $\gamma_0^*(\lambda) = A\gamma_0(\lambda)$, $\gamma_1^*(\lambda) = B\gamma_0(\lambda) + C\gamma_1(\lambda)$, where A, B, C are constants.

Let β denote the largest of the types of the functions $\gamma_0(\lambda)$, $\gamma_1(\lambda)$, and we shall assume that the sequence $\{\lambda_n\}$ is so dense that $\pi\tau > \beta$. We shall show that the system of functions $\{u(\lambda_j, t)\}$, under the additional condition that $\gamma_0^*(\lambda)\gamma_1^*(-\lambda)$ is not an even function (this condition also means that $\gamma_0(\lambda)\gamma_1(-\lambda)$ is not an even function), is complete inside any symmetric domain E' star-shaped with respect to the origin and contained in $T \cap \{|\operatorname{Im}(t)| < \pi\tau - \beta\}$. Having shown this, we thereby prove that the following is valid:

Theorem 1. *The system of solutions $\{y(\lambda_n, z)\}$ of equation (1), under the additional condition that $\gamma_0(\lambda)\gamma_1(-\lambda)$ is not an even function, is complete inside the domain E^*, to which in the t-plane there corresponds the domain E.*

Let $\varphi(\lambda, t)$ denote the solution of the equation $u'' = \lambda^2 u$ satisfying at the point $t = 0$ the same initial conditions as the solution $u(\lambda, t)$ of equation (3). Let d denote some domain in the t-plane, star-shaped with respect to the origin and lying entirely in the domain T. Then, as is known (see, for example, the papers (3)), for any point $t \in d$ the function $u(\lambda, t)$ can be expressed in terms of $\varphi(\lambda, t)$ by means of the transformation operator

$$u(\lambda, t) = \varphi(\lambda, t) + \int_{-t}^{t} K(t, \eta)\varphi(\lambda, \eta)\,d\eta \tag{5}$$

with a kernel $K(t, \eta)$ continuous with respect to η on the interval $[-t, t]$, which depends only on $q(t)$. According to results of V. A. Marchenko (4), the kernel $K(t; \eta)$, when t belongs to a bounded closed set F of the domain d and η varies along the segment $[-t, t]$, remains bounded. Using the initial data for the function $\varphi(\lambda, t)$, we obtain

$$\varphi(\lambda, t) = \frac{\lambda\gamma_0^*(\lambda) + \gamma_1^*(\lambda)}{2\lambda} e^{\lambda t} + \frac{\lambda\gamma_0^*(\lambda) - \gamma_1^*(\lambda)}{2\lambda} e^{-\lambda t}. \tag{6}$$

It follows from relation (6) that $\varphi(\lambda, t)$, as a function of λ, is an entire function of exponential type, and moreover

$$|\varphi(\lambda, t)| < \exp\{|\operatorname{Re}(\lambda t)| + (\beta + \varepsilon)|\lambda|\}, \quad |\lambda| > r_0(\varepsilon). \tag{7}$$

We shall prove that the system of functions $\{\varphi(\lambda_n, t)\}$ is complete inside the horizontal strip $|\operatorname{Im}(t)| < \pi\tau - \beta$.

Let Γ be some rectifiable contour lying entirely in the indicated strip. According to the completeness criterion (see (1), p. 274), in order to prove the theorem it is sufficient to show that from the vanishing of the integrals

$$\int_\Gamma \gamma(t)\varphi(\lambda_n, t)\, dt = 0 \quad (n = 1, 2, ...), \tag{8}$$

where $\gamma(t)$ is a function regular on Γ, outside Γ, and equal to zero at infinity, it follows that $\gamma(t) \equiv 0$. Suppose the equalities (8) hold. Introduce into consideration the function

$$F(\lambda) = \frac{1}{2\pi i} \int_\Gamma \gamma(t)\varphi(\lambda, t)\, dt. \tag{9}$$

Obviously, $F(\lambda_n) = 0$ $(n = 1, 2, ...)$. $F(\lambda)$ is an entire

an exponential-type function, the density of whose positive zeros is not less than τ. From formulas (9) and (7) it follows that $\left|F\left(|\lambda| e^{\pm i\pi/2}\right)\right| < e^{(\pi\tau - \delta - \varepsilon)|\lambda|}$, $|\lambda| > \tau(\varepsilon)$, where δ is the distance between the contour Γ and the boundary of the strip. Hence we conclude that $F(\lambda) \equiv 0$. We shall show that this entails $\gamma(t) \equiv 0$. Denote by $f(\lambda)$ the entire function for which $\gamma(t)$ is the Borel associated function. To prove the theorem, it is plainly sufficient to show that $f(\lambda) \equiv 0$. We have, in accordance with (6) and (9),

$$F(\lambda) = \frac{\lambda\gamma_0^*(\lambda) + \gamma_1^*(\lambda)}{2\lambda} f(\lambda) + \frac{\lambda\gamma_0^*(\lambda) - \gamma_1^*(\lambda)}{2\lambda} f(-\lambda) \equiv 0,$$

$$F(-\lambda) = \frac{-\lambda\gamma_0^*(-\lambda) + \gamma_1^*(-\lambda)}{-2\lambda} f(-\lambda) + \frac{-\lambda\gamma_0^*(-\lambda) - \gamma_1^*(-\lambda)}{-2\lambda} f(\lambda) \equiv 0.$$

Hence we conclude that, since $\gamma_0^*(\lambda)\gamma_1^*(-\lambda) \not\equiv \gamma_0^*(-\lambda)\gamma_1^*(\lambda)$, we have $f(\lambda) \equiv 0$.

For the proof of theorem 1 we turn to the operator (5). A. F. Leont' ev [3] showed that if the coefficient $q(t)$ in equation (3) and the function $\varphi(\lambda, t)$ are regular in a domain B, symmetric and star-shaped with respect to the origin, then the function $u(\lambda, t)$, defined by the operator (5), is also regular inside B; moreover, the converse assertion is also true, i.e., if the function $u(\lambda, t)$ is regular in the indicated domain B, then the function $\varphi(\lambda, t)$ is also regular in it. In addition, the operator (5) is a continuous operator and has in the domain B a continuous inverse operator. Let E be an arbitrary closed bounded domain, symmetric and star-shaped with respect to the origin, contained in $T \cap \{|\operatorname{Im}(z)| < \pi\tau - \beta\}$, and let $\varphi(t)$ be an arbitrary analytic function in E. Apply to $\varphi(t)$ the operator inverse to the operator (5). We thereby obtain a certain function $\psi(t)$, regular in E. The function $\psi(t)$ can be uniformly approximated in E by linear combinations of functions of the system $\{\varphi(\lambda_n, t)\}$:

$$|\psi_n(t)| = \left|\psi(t) - \sum_{k=1}^{p_n} \alpha_k^{(n)} \varphi(\lambda_k, t)\right| < \varepsilon_n, \qquad t \in E, \quad \varepsilon_n \to 0, \quad n \to \infty.$$

Apply the operator (5) to the functions $\psi_n(t)$ $(n = 1, 2, \dots)$. By virtue of the continuity of this operator we shall have

$$\left|\varphi(t) - \sum_{k=1}^{p_n} \alpha_k^{(n)} u(\lambda_k, t)\right| < \varepsilon_n (1 + P_E), \qquad t \in E,$$

where P_E is a constant depending on the domain E. Thus theorem 1 is proved.

The use of the transformation operator (5) makes it possible to transfer a number of results concerning sequences of Dirichlet polynomials to the sequences (2).

From formulas (5) and (6), for the functions of the sequence

$$Q_n^*(t) = \sum_{j=1}^{q_n} \alpha_j^{(n)} u(\lambda_j, t) \qquad (n = 1, 2, \dots) \tag{10}$$

we obtain the expressions

$$Q_n^*(t) = P_n(t) + \int_{-t}^{t} K(t, \eta) P_n(\eta)\, d\eta \qquad (n = 1, 2, \dots), \tag{11}$$

where $P_n(t)$ are Dirichlet polynomials:

$$P_n(t) = \sum_{j=1}^{q_n} \alpha_j^{(n)} \left[A_j e^{\lambda_j t} + B_j e^{-\lambda_j t}\right] \qquad (n = 1, 2, \dots). \tag{12}$$

(A_j and B_j are certain constants). Denote by g a domain in D_0 which, under the transformation (5), passes into a domain g_0 containing within itself a closed vertical segment R of length $2\pi\tau$ with center at the origin. Suppose that the sequence (2) converges uniformly in the domain g_0. Then the sequence (10) will converge uniformly in the domain g_0. According to the properties of the operator (5) listed above, in some domain containing R the sequence of Dirichlet polynomials (12) will converge uniformly. But then the sequence (12) will converge, as A. F. Leont' ev[5] showed, in some vertical strip $\beta_1 < \mathrm{Re}(t) < \beta_2$.

Let $G \in T$ be a domain, symmetric and star-shaped with respect to the origin, which is contained in the indicated strip. In this domain the sequence (10) will converge uniformly. In the z-plane there corresponds to the domain G a certain domain F; in it the sequence (2) will converge uniformly. Thus we have shown that the following holds:

Theorem 2. *A sequence (2) that converges uniformly in a domain g which contains within itself an arc l_τ, passing under the mapping (5) into a vertical segment of length $2\pi\tau$, converges uniformly inside the domain F.*

Just as for Dirichlet polynomials, under the conditions of Theorem 2 the limits $\lim_{n\to\infty} a_j^{(n)} = a_j$ exist, and the uniqueness theorem holds: two sequences of the form (2) converge to one and the same function if and only if the limits of the corresponding coefficients are equal.

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Note: Figure translations are in progress. See original paper for figures.