HYDROMECHANICS

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Abstract

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ON THE PROBLEM OF THE FLOW OF AN IDEAL INCOMPRESSIBLE FLUID THROUGH A GIVEN DOMAIN*

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The velocity $\mathbf{v}(x,t)$ and the pressure $P(x,t)$ in the case of an ideal incompressible fluid satisfy the system of Euler equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla P + \mathbf{F}(x,t), \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{2}$$

which must hold at each instant of time t and at each point x of the flow domain Ω. We shall assume that the flow at the initial instant is known:

$$\mathbf{v}|_{t=0} = \mathbf{a}(x). \tag{3}$$

If the boundary S of the domain is an impermeable wall, then on it $v_n = \mathbf{v}\cdot\mathbf{n} = 0$ ($\mathbf{n}$ is the outward normal). If the boundary S is permeable, then it is natural to impose the boundary condition

$$v_n|_S = \gamma(x,t). \tag{4}$$

In the case $\gamma \equiv 0$, conditions (1)–(4) determine a unique fluid motion. "In the small" this was established in ([1,2]), and for the case of two-dimensional motions "in the large" in ([3]). If, however, $\gamma \not\equiv 0$, these conditions are insufficient for the unambiguous determination of the solution, and one has to impose one more boundary condition. Let us formulate it.

Let S_t^- be the part of the boundary S on which $\gamma(x,t) < 0$; S_t^+ is the remaining part. The indicated condition has the form: for any $t > 0$,

$$\operatorname{rot} \mathbf{v}|_{S_t^-} = \vec{\pi}(x,t), \tag{5}$$

where $\vec{\pi}(x,t)$ is an arbitrarily prescribed vector. This boundary condition in one particular case was proposed by N. E. Kochin in ([4]).

Below we shall present the course of the proof of existence and uniqueness of a classical solution of problem (1)–(5) in the large for the case of two-dimensional motions**. We shall assume that the following conditions are satisfied.

A. The flow domain Ω is finite; S consists of $n+1$ closed contours $S_0, S_1, \dots, S_n$ (S_0 encloses the others) of class $C^{(4)}$.

B. There exists a vector $\mathbf{b}(x,t)$, defined in the cylinder $Q_T = \Omega \times [0,T]$ ($T>0$ is any fixed number in what follows), such that $\mathbf{b} \in W_p^{(3)}(\Omega)$, $\partial \mathbf{b}/\partial t \in W_p^{(2)}(\Omega)$ ($p>2$) and the corresponding norms are bounded for $t \in [0,T]$, moreover

$$\operatorname{div} \mathbf{b} = 0; \qquad \mathbf{b}|_{t=0} = \mathbf{a}; \qquad \mathbf{b}\cdot\mathbf{n}|_S = \gamma; \qquad \omega_0(x,t)|_{S_t^-} = \pi(x,t), \tag{6}$$

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** For the three-dimensional problem, analogous considerations lead only to a proof of solvability locally in t, since here we do not know an estimate for $\max|\operatorname{rot}\mathbf{v}|$.

where ω_0 is the curl of the vector $\mathbf{b}$.* This assumption replaces the compatibility conditions.

C. S_t^- consists of several contours S_k, and hence, on S_t^-,

$$|\gamma| \geqslant c > 0. \tag{7}$$

If this condition is abandoned, then one has to impose certain new conditions at the junction points of S_t^- and S_t^+.

D. The external force $\mathbf{F}(x,t)$ is a vector of class $C^{(1)}(Q_T)$; its curl $f(x,t)$ has continuous derivatives $\partial f/\partial x_k$ and bounded $\partial^2 f/\partial x_i \partial x_k$.

E. For $t=0$, on S_0^- the equality

$$\frac{\partial \pi}{\partial t} + \mathbf{a}\operatorname{grad}\omega_0 = f \tag{8}$$

holds.

This is a necessary condition for continuity of the first derivatives of the velocity curl $\omega(x,t)$ in Q_T. The velocity curl $\omega(x,t)$ satisfies the equation

$$\frac{\partial \omega}{\partial t} + v_k \frac{\partial \omega}{\partial x_k} = f. \tag{9}$$

Theorem. *Suppose that conditions A—E are satisfied. Then there exists, and moreover only one**, solution $\mathbf{v}(x,t), P(x,t)$ of problem (1)—(5). In addition, all derivatives occurring in relations (1)—(5) and (9) are continuous in Q_T, and these relations are satisfied in the classical sense.*

Existence is proved by means of the following iterative process***. As the zeroth approximation for the velocity we take the vector $\mathbf{b}(x,t)$. Suppose that the $(m-1)$-st approximation $\mathbf{v}_{m-1}(x,t)$ for the velocity has already been found. First define the m-th approximation for the curl $\omega_m(x,t)$ as the solution of the linear problem

$$\frac{\partial \omega_m}{\partial t} + \mathbf{v}_{m-1} \cdot \operatorname{grad} \omega_m = f(x,t); \tag{10}$$

$$\omega_m = \omega_0(x,t) \tag{11}$$

for $t=0$ and on S_t^-. After this we find the m-th approximation for the velocity $\mathbf{v}_m(x,t)$:

$$\operatorname{rot} \mathbf{v}_m = \vec{\omega}_m(0,0,\omega_m), \tag{12}$$

$$\operatorname{div} \mathbf{v}_m = 0, \tag{13}$$

$$\mathbf{v}_m \cdot \mathbf{n}\big|_S = \gamma, \tag{14}$$

$$\Gamma_{mk} \equiv \oint_{S_k} \mathbf{v}_m \cdot \mathbf{s}\, ds = \oint_{S_k} \mathbf{a} \cdot \mathbf{s}\, ds + \int_0^t \oint_{S_k} \mathbf{F}(x,\tau) \cdot \mathbf{s}\, ds\, d\tau - \int_0^t \oint_{S_k} \gamma \omega_m\, ds\, d\tau$$

$$(k = 1, 2, \dots, n). \tag{15}$$

Problem (10)—(11) can be solved on the basis of simple hydrodynamical considerations. Suppose that some fluid particle at the moment t is located at the point $x \in \Omega$. Define the trajectory of its preceding motion:

$$\frac{d\alpha}{d\tau} = \mathbf{v}_{m-1}(\alpha,\tau), \qquad \alpha\big|_{\tau=t} = x, \qquad \alpha = (\alpha_1, \alpha_2). \tag{16}$$

* In view of the two-dimensionality of the problem, only one component of the curl is nonzero; we call it the curl.

** Of course, the pressure P is determined up to an arbitrary function of time.

*** We note that this process can be adapted for approximate computation of the solution.

(16) determines the vector-function $\alpha = \alpha(x,t,\tau)$. Two cases are possible: either for all $\tau \in [0,t]$ the point $\alpha(x,t,\tau)$ lies inside the domain Ω, or at some moment $\tau'(x,t)$ $(0 \le \tau' \le t)$ $\alpha \in S_{\tau'}$. Define the function $\tau^*(x,t)$, setting it in the first case equal to zero, and in the second to $\tau'(x,t)$. Introduce the vector-function $\alpha^*(x,t) = \alpha(x,t,\tau^*(x,t))$. Obviously, α^*, τ^* are, respectively, the place and time of appearance of the particle under consideration in the flow domain Ω. The solutions of (16) are the characteristics of equation (10). We have:

$$\omega_m(x,t) = \omega_0(\alpha^*(x,t), \tau^*(x,t)) + \int_{\tau^*(x,t)}^{t} f(\alpha(x,t,\tau),\tau)\, d\tau. \tag{17}$$

In particular, when $f = 0$, equality (17) expresses the fact that the vortex ω_m does not change along the trajectory joining (α^*, τ^*) and (x,t).

We seek the solution of problem (12)–(15) in the form

$$\mathbf{v}_m = \operatorname{grad}\Phi(x,t) + \mathbf{u}_{0m}(x,t) + \sum_{k=1}^{n} \lambda_{km}(t)\mathbf{u}_k(x), \tag{18}$$

where Φ is the solution of the Neumann problem

$$\Delta\Phi = 0; \qquad \left.\frac{\partial\Phi}{\partial n}\right|_S = \gamma(x,t), \tag{19}$$

the vectors $\mathbf{u}_{0m}, \mathbf{u}_k$ have stream functions ψ_{0m}, ψ_k, determined by solving the problems

$$\Delta\psi_k = 0; \qquad \psi_k|_{S_i} = \delta_{ik} \qquad (k = 1,2,\dots,n;\ i = 0,1,\dots,n); \tag{20}$$

$$\Delta\psi_{0m} = -\omega_m; \qquad \psi_{0m}|_S = 0;$$

$$\lambda_{km}(t) = \sum_{i=1}^{n} \xi_{ik} \left(\Gamma_{mi} - \int_\Omega \psi_i \omega_m\, dx + \int_\Omega \mathbf{u}_{0m} \cdot \mathbf{u}_i\, dx \right), \tag{21}$$

where ξ_{ik} are known constants depending only on the domain Ω.

Lemma 1. The unique solution of problem (10)–(11) is determined by formula (17), and the first derivatives of $\omega_m(x,t)$ are continuous in $\overline{Q_T}$, while the second derivatives $\partial^2\omega_m/\partial x_i \partial x_k$, $\partial^2\omega_m/\partial x_i \partial t$ are bounded.

Lemma 2. Problem (12)–(15) has, and moreover has a unique, solution $\mathbf{v}_m(x,t)$ such that $\mathbf{v}_m \in W_p^{(3)}(\Omega)$, $\partial \mathbf{v}_m/\partial t \in W_p^{(2)}(\Omega)$, and

$$\|\mathbf{v}_m(x,t)\|_{W_r^{(2)}(\Omega)} + \left\|\frac{\partial \mathbf{v}_m}{\partial t}\right\|_{W_r^{(1)}(\Omega)} \leq C_1 r \left[\|\omega_m\|_{W_r^{(1)}(\Omega)} + \left\|\frac{\partial \omega_m}{\partial t}\right\|_{L_r(\Omega)}\right] + C_2, \quad (22)$$

where $r \geq 2$ is arbitrary, and C_1, C_2 do not depend on r, m.

In the proof, the result from ([5]) is used.

Lemma 3. The following estimates, uniform in m, are valid ($r > 1$):

$$\max_{x,t} |\omega_m(x,t)| < C_3; \qquad \max_t \left\{\|\nabla \omega_m\|_{L_r(\Omega)} + \left\|\frac{\partial \omega_m}{\partial t}\right\|_{L_r(\Omega)}\right\} < C_4. \quad (23)$$

The first of estimates (23) is obtained by means of a device generalizing that used for the same purpose in ([3]); the derivation of the second estimate is based on the following lemma.

Lemma 4. Let $u(x)$ be the solution of the boundary-value problem

$$\Delta u = f(x); \qquad u|_S = 0 \quad (24)$$

in a bounded k-dimensional domain Ω with boundary $S \in C^{(3)}$. Let $|f| \leq M$ and $f \in W_p^{(1)}$ ($p > k$). Then the estimate

$$|D^2 u| \leq M_1 \ln \left[M_2 + M_3 \|\nabla f\|_{L_p(\Omega)}\right], \quad (25)$$

is valid, where M_1, M_2, M_3 depend only on M, S, p.

With the aid of Lemma 3 it is shown that the sequence $\{\omega_m\}$, as $m \to \infty$, converges strongly in any $L_r(\Omega)$, uniformly in $t \in [0,T]$, to some function $\omega(x,t)$, while the first derivatives converge weakly in any $L_r(Q_T)$; the sequence $\{\mathbf{v}_m\}$ converges in $W_r^{(1)}(\Omega)$, uniformly in t, to a vector $\mathbf{v}(x,t)$. This makes it possible to pass to the limit as $m \to \infty$ in (10)–(15).

Next it is shown that the limiting system (obtained by dropping the indices $m, m-1$) is equivalent to the original problem (1)–(5). We note that the limiting equality for (15) is equivalent to the uniqueness condition for the pressure.

The preceding discussion admits a generalization to the case of a domain deforming in a prescribed manner with time. It is also not difficult to carry out a further investigation of the differential properties of the solution.

In conclusion, let us note that it would be interesting to consider, instead of (5), other variants of the additional boundary condition, as well as the corresponding stationary problems.

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Note: Figure translations are in progress. See original paper for figures.