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Abstract

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ON FINITE Π-CONJUGACY OF GROUPS

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§ 1. The study of conditions under which all Sylow Π-subgroups (Π an arbitrary set of prime numbers) of a given group are conjugate to one another constitutes an important direction in modern group theory. The series of works of S. A. Chunikhin, begun with the introduction of the concept of a Π-separable group ([1]), determined the development of this direction (both in our country and abroad) in the theory of finite groups (see ([2])). The theorem of A. P. Ditsman, A. G. Kurosh, and A. I. Uzkov (([3]), § 54), repeatedly generalized by various authors, laid the foundation for the successful development of the same direction in the theory of infinite groups. The present note is devoted to questions of conjugacy of Π-Sylow subgroups in infinite groups.

§ 2. **Definition.** We shall say that a group $\mathfrak{G}$ possesses the property of **finite Π-conjugacy** if from the existence in it of a finite class of conjugate Π-Sylow subgroups it follows that all its Π-Sylow subgroups are conjugate.

Lemma. *A group $\mathfrak{G}$ possesses the property of finite Π-conjugacy if all subgroups of finite order in it itself and in any of its factor groups possess the property of conjugacy of Π-Sylow subgroups.*

Proof. Let $\mathfrak{G}$ contain a finite class of conjugate Π-Sylow subgroups:

$$\mathfrak{M},\ c_2^{-1}\mathfrak{M}c_2, \dots,\ c_\lambda^{-1}\mathfrak{M}c_\lambda. \tag{1}$$

To this class there corresponds the likewise finite class of normalizers:

$$N_{\mathfrak{M}},\ c_2^{-1}N_{\mathfrak{M}}c_2, \dots,\ c_\lambda^{-1}N_{\mathfrak{M}}c_\lambda. \tag{2}$$

Denote the intersection of all subgroups of class (1) by D, and that of class (2) by σ. It is clear that the subgroups D and σ are invariant in $\mathfrak{G}$. The subgroup σ, moreover, by the well-known theorem of Poincaré, has finite index in $\mathfrak{G}$. In the case $\sigma = 1$, by the hypothesis, the lemma is true. We shall therefore suppose that $\sigma \neq 1$. It is easy to see that D is the unique Π-subgroup in σ, for otherwise the subgroup σ would contain some Π-element h not belonging to D and not belonging to at least one of the subgroups $c_k^{-1}\mathfrak{M}c_k$ of class (1), but permutable with it. But this would contradict the fact that the subgroups of class (1) are Π-Sylow, since the Π-subgroup $\{h\}c_k^{-1}\mathfrak{M}c_k$, distinct from $c_k^{-1}\mathfrak{M}c_k$, would contain the latter. Consider the factor group $\mathfrak{G}/D$. To each subgroup $c_i^{-1}\mathfrak{M}c_i$ in $\mathfrak{G}$ belonging to class (1), there corresponds the Π-Sylow subgroup $\frac{c_i^{-1}\mathfrak{M}c_i}{D}$ in $\mathfrak{G}/D$. From the isomorphism theorem it follows that this subgroup is finite:

$$\frac{c_i^{-1}\mathfrak{M}c_i\sigma}{\sigma} \cong \frac{c_i^{-1}\mathfrak{M}c_i}{D}.$$

Thus, to a finite class of Π-Sylow subgroups in $\mathfrak{G}$ there corresponds a finite class of finite Π-Sylow sub-

groups in $\frac{\mathfrak{G}}{D}$:

$$\frac{\mathfrak{M}}{D}, \quad \frac{c_2^{-1}\mathfrak{M}c_2}{D}, \dots, \frac{c_\lambda^{-1}\mathfrak{M}c_\lambda}{D}. \tag{3}$$

The totality of all distinct elements (adjacent systems) from subgroups of class (3) may be regarded as a finite invariant set of elements of finite order (the case $D = 1$ does not contradict this conclusion). Therefore, by A. P. Ditsman' s lemma ([(3)], § 53), the totality of subgroups of class (3) generates a finite subgroup $\frac{H}{D}$, invariant in $\frac{\mathfrak{G}}{D}$. Let now M be any Π-Sylow subgroup in $\mathfrak{G}$. Then, for the reason indicated above, $\frac{M}{D}$ is a finite Π-Sylow subgroup in $\frac{\mathfrak{G}}{D}$. But then the subgroup $\frac{M}{D} \cdot \frac{H}{D}$ must also be finite (its coincidence with the whole group $\frac{\mathfrak{G}}{D}$ is not excluded). By the hypothesis of the lemma, in this group all Π-Sylow subgroups are conjugate, but it contains $\frac{\mathfrak{M}}{D}$ and $\frac{M}{D}$. Consequently, $\frac{M}{D} = s^{-1}D\frac{\mathfrak{M}}{D}sD = \frac{s^{-1}\mathfrak{M}s}{D}$, where $s \in M \cdot H \subseteq \mathfrak{G}$, and this means that $M = s^{-1}\mathfrak{M}s$, with $s^{-1}\mathfrak{M}s$ being a subgroup of class (1). The lemma is proved.

Below we give some propositions following from the lemma just proved.

§ 3. First of all, it is obvious that the lemma includes the main assertion of the theorem of Ditsman–Kurosh–Ushakov noted above and thus gives a new proof of it, since in the case when the set Π consists of one prime number p occurring in the orders of the elements of the group, the condition of the lemma is, in view of Sylow' s theorem, satisfied for any group. Therefore the following holds:

Theorem 1 (Ditsman–Kurosh–Ushakov). *An arbitrary group has the property of finite p-conjugacy.*

§ 4. In our note ([4]) one of the generalizations of the well-known Hall theorem on finite solvable groups ([5]) was given. Hall' s theorem was extended there to arbitrary periodic solvable groups, i.e. periodic groups possessing a finite normal series with abelian factors. However, the lemma proved now makes it possible to remove one of the restrictions of that theorem, namely the condition of periodicity of the group. We have:

Theorem 2. *An arbitrary solvable group has the property of finite* Π*-conjugacy.*

§ 5. A further generalization of Theorem 2 arises on passing to generalized solvable groups ([3], § 57). Indeed, the condition of the lemma is valid for $R\bar{N}$-, RJ-, RN^*- and RJ^*-groups, since, considering all these groups as RN-groups, one may conclude that every finite subgroup in any of them is also an RN-group and, consequently, solvable. On the other hand, the indicated classes of groups are closed with respect to taking factor groups. This means that every finite subgroup in any of their factor groups is solvable. Hence it follows:

Theorem 3. *Generalized solvable groups of the types* $R\bar{N}$*,* RJ*,* RN^* *and* RJ^* *have the property of finite* Π*-conjugacy.*

The question of the validity of the theorem for groups of the types RN, RJ and RK remains open for the time being.

§ 6. P. A. Gol' berg, following S. A. Chunikhin, considers arbitrary Π-separable groups, defining them as groups possessing an ascending normal series in which the orders of the elements of any factor may contain not more than one prime number from Π ([6]). Such

groups are closed with respect to taking subgroups, and, under the additional condition of periodicity, they are also closed with respect to taking factor groups. As for the finite Π-separable groups of S. A. Chunikhin, by Hall' s embedding theorem (see, for example, (7), p. 39), they satisfy the condition of conjugacy of Π-Sylow subgroups. The requirements of the lemma are thus fulfilled and, consequently, the following may be formulated:

Theorem 4. *A periodic* Π*-separable group has the property of finite* Π*-conjugacy.*

§ 7. In the note (8) we proved a local theorem for the property of finite Π-conjugacy. From it followed the property of finite Π-conjugacy for locally finite and locally soluble groups. Now, thanks to Theorem 2, we can dispense with the condition of local finiteness.

Theorem 5. *A locally soluble group has the property of finite* Π*-conjugacy.*

If by a locally Π-separable group one understands a group each finite subset of whose elements generates a finite Π-separable group (6), then from the same local theorem (8) it follows:

Theorem 6. *A locally* Π*-separable group has the property of finite* Π*-conjugacy.*

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Note: Figure translations are in progress. See original paper for figures.