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MATHEMATICS

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1962

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Abstract

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MATHEMATICS

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SOME COVERING THEOREMS FOR UNIVALENT FUNCTIONS

(Presented by Academician V. I. Smirnov, 3 VIII 1961)

Let $S(a_1, a_2)$ be the class of functions $w = f(z)$, $f(0) = 0$, regular and univalent in the disk $|z| < 1$ and mapping it onto domains containing no points a_1, a_2; $f(z; a_1, a_2)$ is the unique function of this class for which $|f'(0)| \le f'(0; a_1, a_2)$, $f(z) \in S(a_1, a_2)$*. Let S be the class of functions $w = f(z) = z + c_2 z^2 + \cdots$, regular and univalent in the disk $|z| < 1$; S_R is the subclass of functions of the class S with real coefficients.

We shall denote, as usual, by K the complete elliptic integral of the first kind with modulus k, and by $\operatorname{sn} z$ and $\theta(z)$ the Jacobi elliptic functions with the same modulus k.

Theorem 1. If the function $f(z) = cz + \cdots \in S(a_1, a_2)$, $|a_1| = |a_2| = a$, $|\arg a_1 - \arg a_2| = 2\alpha$ $(0 \le \alpha \le \pi/2)$, then the sharp inequality is valid

$$\frac{a}{|c|} \ge h(\alpha) = \begin{cases} \dfrac{1}{4}, & \alpha = 0, \\ \left\{ \sqrt{\dfrac{4pm + (m^2 - p^2)^2}{16m^2} \, \dfrac{\theta(0)}{\theta(2u)}} \right\}, & 0 < \alpha < \dfrac{\pi}{2}, \\ \dfrac{1}{2}, & \alpha = \dfrac{\pi}{2}, \end{cases} \tag{1}$$

where the functions $u = u(\alpha)$, $m = m(\alpha)$, $p = p(\alpha)$ and $k = k(\alpha)$ are uniquely determined for $0 < \alpha < \pi/2$ by the relations

$$\operatorname{sn} u = m - p \quad (0 < u < K), \qquad \frac{m\cos\alpha - 1}{\sqrt{pm}} = \frac{\theta'(u)}{\theta(u)},$$

$$p = \sqrt{m^2 - 2m\cos\alpha + 1}, \qquad k^2 = \frac{p + m - \cos\alpha}{2p}.$$

Equality in (1) occurs for $0 \le \alpha \le \pi/2$ only for the functions $f(z) = f(\varepsilon z; a_1, a_2)$, $|\varepsilon| = 1$. For $0 < \alpha < \pi/2$ each of the indicated extremal functions maps the disk $|z| < 1$ onto the entire w-plane with a single slit consisting of the ray

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$\arg w = \arg\sqrt{a_1 a_2}$, $|w| \geq ma$ $(1 < m < \infty)$, and two circular arcs symmetric with respect to the straight line containing this ray, with ends at the points a_1, $m\sqrt{a_1 a_2}$ and a_2, $m\sqrt{a_1 a_2}$. For $\alpha = 0$ and $\alpha = \pi/2$ the extremal functions

$$\left(f(\varepsilon z; a_1, a_1) = \frac{4a_1 \varepsilon z}{(1+\varepsilon z)^2}, \quad f(\varepsilon z; a_1, -a_1) = \frac{2a_1 \varepsilon z}{1+\varepsilon^2 z^2}, \quad |\varepsilon| = 1 \right)$$

map the disk $|z| < 1$ onto the entire w-plane with a single radial slit $\arg w = \arg a_1$, $|w| \geq a$ in the first case, and with two radial slits $\arg w = \arg a_1$, $|w| \geq a$ and $\arg w = \arg a_2$, $|w| \geq a$ in the second case**.

The proof of Theorem 1 is based on the use of the known properties of the function $w = f(z)$ realizing $\max|f'(0)|$ in the given class, and ana-

* The points a_1, a_2 may be either distinct or coincident.
** Here and below $\arg\sqrt{a_1 a_2}$ corresponds to the bisector of the smaller angle determined by $\arg a_1, \arg a_2$.

···analytic expression for the inverse function $z = f^{-1}(w)$, obtained by M. A. Lavrent' ev (1) on the basis of the principles of Lindelöf and Montel, whence there also follows the uniqueness of the extremal function $w = f(z)$, normalized by the condition $f'(0) > 0$. In the limiting cases $\alpha = 0$ and $\alpha = \pi/2$ the assertion of the theorem constitutes the known results of Koebe and Szegő for an unnormalized univalent mapping.

Remark. For $0 < \alpha < \pi/2$ the analytic expression for the inverse function $z = f^{-1}(w)$, where $w = f(z)$ is the extremal function of Theorem 1, is determined, up to a rotation, by the quantities $h(\alpha)$ and $m(\alpha)$ obtained in Theorem 1 and having the geometric meaning indicated there.

Corollary. If the function $f(z) = z + c_2 z^2 + \cdots \in S$ does not assume in the disk $|z| < 1$ the values a_1 and a_2, $|a_1| = |a_2| = a$, $|\arg a_1 - \arg a_2| = 2\alpha$ $(0 \leq \alpha \leq \pi/2)$, then we have the sharp inequality

$$a \geq h(\alpha) \qquad \left(0 \leq \alpha \leq \frac{\pi}{2}\right),$$

and the equality sign is realized only by the functions $f(z) = f(z; a_1, a_2)$.

For $0 < \alpha \leq \pi/2$ this corollary geometrically means that the image of the disk $|z| < 1$ under the mapping by any function $w = f(z)$ of the class S contains at least one of the points, equidistant from the origin, of any two segments of length $h(\alpha)$, issuing from the point $w = 0$ at an angle α to one another; and the ends of both these segments do not belong to the image of the disk $|z| < 1$ only for the indicated extremal functions. For $\alpha = 0$ we obtain the well-known Koebe theorem on covering the disk $|w| < 1/4$ by the image of the unit disk under its mapping by any function of the class S.

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Denote by $f(|z|<1)$ the image of the disk $|z|<1$ under the mapping $w=f(z)$, and by $\overline{f(|z|<1)}_{\beta}$ the domain symmetric to $f(|z|<1)$ with respect to the straight line containing the ray $\arg w=\beta$.

From the corollary to Theorem 1 one easily obtains

Theorem 2. The set

$$\bigcap_{f\in S}\left[f(|z|<1)\cup\overline{f(|z|<1)}_{\beta}\right]$$

of points $w=\rho e^{i\varphi}$ is a domain containing the point $w=0$, bounded by the curve $w=R_{\beta}(\varphi)e^{i\varphi}$, where $R_{\beta}(\varphi)=h(|\varphi-\beta|)$, $-\pi/2+\beta\le\varphi\le\pi/2+\beta$, $R_{\beta}(\varphi)=h(|\varphi-\beta-\pi|)$, $\pi/2+\beta<\varphi<3\pi/2+\beta$.

The point $R_{\beta}(\varphi)e^{i\varphi}$ belongs to neither of the domains $f(|z|<1)$ and $\overline{f(|z|<1)}_{\beta}$ only for the functions

$$f(z)=f(z;R_{\beta}(\varphi)e^{i\varphi},\ R_{\beta}(\varphi)e^{i(-\varphi-2\beta)})$$

when $|\varphi-\beta|\le\pi/2$, and

$$f(z)=f(z;R_{\beta}(\varphi)e^{i\varphi},\ R_{\beta}(\varphi)e^{i(-\varphi+2\beta+2\pi)})$$

when $|\varphi-\beta-\pi|<\pi/2$.

A consequence of Theorem 2 is

Theorem 3. The set

$$\bigcap_{f\in S_R^*}f(|z|<1)$$

of points $w=\rho e^{i\varphi}$ is a domain containing the point $w=0$, bounded by the curve $w=R(\varphi)e^{i\varphi}$, where $R(\varphi)=h(|\varphi|)$, $-\pi/2\le\varphi\le\pi/2$, $R(\varphi)=h(|\varphi-\pi|)$, $\pi/2<\varphi<3\pi/2$.

The point $R(\varphi)e^{i\varphi}$ does not belong to the domain $f(|z|<1)$ only for the functions

$$f(z)=f(z;R(\varphi)e^{i\varphi},\ R(\varphi)e^{-i\varphi})$$

when $|\varphi|\le\pi/2$, and

$$f(z)=f(z;R(\varphi)e^{i\varphi},\ R(\varphi)e^{i(-\varphi+2\pi)})$$

when $|\varphi-\pi|<\pi/2$.

Remark. The domain considered in Theorem 3, without a concrete determination of its boundary and of the functions $f(z)$ for which the points of this boundary do not belong to the domain $f(|z|<1)$, was obtained by Jenkins

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([2]) as one of the examples of the systematic use of the method of the extremal metric and the theory of quadratic differentials.

Let $S^*(a_1, a_2)$ and S^* be the sets of those functions of the classes $S(a_1, a_2)$ and S, respectively, which map the disk $|z| < 1$ onto domains star-shaped with respect to the origin.

From Lindelöf' s principle it easily follows that

Theorem 4. If the function $f(z) = cz + \cdots \in S^*(a_1, a_2)$, $|a_1| = |a_2| = a$, $|\arg a_1 - \arg a_2| = 2\pi\lambda$ $(0 \le \lambda \le 1/2)$, then $\cdots$

the sharp inequality

$$\frac{a}{|c|} \ge \frac{1}{4}\lambda^{-\lambda}(1-\lambda)^{-(1-\lambda)} \qquad (0 \le \lambda \le 1/2),$$

and the equality sign is realized only by the functions

$$f(z) = \frac{4\lambda^{\lambda}(1-\lambda)^{1-\lambda}\sqrt{a_1 a_2}\varepsilon z}{(1-\varepsilon z)^{2\lambda}(1+\varepsilon z)^{2(1-\lambda)}}, \qquad |\varepsilon| = 1.$$

For $0 < \lambda \le 1/2$ the indicated extremal functions map the disk $|z| < 1$ onto the entire w-plane with two radial slits $\arg w = \arg a_1$, $|w| \ge a$, and $\arg w = \arg a_2$, $|w| \ge a$. For $\lambda = 0$ the extremal functions map the disk $|z| < 1$ onto the entire w-plane with a single radial slit $\arg w = \arg a_1$, $|w| \ge a$.

Corollary. If the function $f(z) = z + c_2 z^2 + \ldots \in S^*$ does not assume in the disk $|z| < 1$ the values a_1 and a_2, $|a_1| = |a_2| = a$, $|\arg a_1 - \arg a_2| = 2\pi\lambda$ $(0 \le \lambda \le 1/2)$, then the sharp inequality

$$|a| \ge \frac{1}{4}\lambda^{-\lambda}(1-\lambda)^{-(1-\lambda)} \qquad (0 \le \lambda \le 1/2),$$

holds, and the equality sign occurs only for the functions

$$f(z) = \frac{z}{(1 - e^{-i\arg\sqrt{a_1 a_2}}z)^{2\lambda}(1 + e^{-i\arg\sqrt{a_1 a_2}}z)^{2(1-\lambda)}}.$$

This corollary makes it possible to obtain for the class S^* certain covering theorems, in the same way as the corollary to Theorem 1 gives the possibility of obtaining covering theorems for the class S.

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Received 2 VIII 1961

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Note: Figure translations are in progress. See original paper for figures.

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