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Abstract

Full Text

Mathematics

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER IN A BANACH SPACE WHEN THE SMALL PARAMETER AT THE FIRST AND SECOND DERIVATIVES TENDS TO ZERO

(Presented by Academician S. L. Sobolev on 23 X 1961)

Consider, in a Banach space Y , the ordinary linear homogeneous differential equation of second order

$$\mu y'' + \mu p(x)y' = y \tag{1}$$

with a small parameter $\mu > 0$ tending to zero. The unknown function $y = y(x, \mu) \in Y$ is a function of the real variable x ; $p(x)$ is a given function of the variable x with values in the space of linear bounded operators $\{Y \rightarrow Y\}$ ⁽¹⁾, mapping the space Y into Y . We study the behavior of any solution of equation (1) with Cauchy initial data as μ tends to zero. We note that the limiting equation for equation (1) has the solution $y_0(x) \equiv 0$.

Assume that the function $p(x)$ is continuous, infinitely differentiable, and

$$\|p(x)\| \leq K, \quad \|p^{(n)}(x)\| \leq M(n) \tag{2}$$

for $x_1 \leq x \leq x_2$, where $K, M(n)$ are constants ($M(n)$ may increase together with n). Under these assumptions it will be proved that the norm of any solution $y(x, \mu)$ of equation (1), as $\mu \rightarrow 0$, tends to infinity of order $\exp(1/\mu)$. The behavior of the derivative $y'(x, \mu)$ is analogous.

Putting $\mu = \varepsilon^2$ in (1), we shall have

$$\varepsilon^2 y'' + \varepsilon^2 p(x)y' = y. \tag{3}$$

Simultaneously with equation (3), consider the equation ⁽²⁾

$$\varepsilon^2 \bar{y}'' + \varepsilon^2 p(x) \bar{y}' = \bar{y}. \quad (4)$$

in the space $\{Y \rightarrow Y\}$.

Theorem 1. Equation (4) has two independent solutions $\bar{y}_1(x, \varepsilon)$, $\bar{y}_2(x, \varepsilon)$, satisfying the asymptotic relations

$$\bar{y}_1(x, \varepsilon) \sim \exp\left(-\frac{x-x_0}{\varepsilon}\right) \sum_{i=0}^{\infty} \bar{u}_{1i}(x) \varepsilon^i, \quad (5)$$

$$(x_1 \leq x_0 \leq x \leq x_2)$$

$$\bar{y}_2(x, \varepsilon) \sim \exp\left(\frac{x-x_0}{\varepsilon}\right) \sum_{i=0}^{\infty} \bar{u}_{2i}(x) \varepsilon^i, \quad (6)$$

where the functions $\bar{u}_{1i}(x)$, $\bar{u}_{2i}(x) \in \{Y \rightarrow Y\}$ are continuous, infinitely differentiable, and moreover

$$\bar{y}_1(x_0, \varepsilon) = I, \quad \bar{y}_2(x_0, \varepsilon) = I \quad (7)$$

(I is the identity operator).

The solutions \bar{y}_1 , \bar{y}_2 are independent in the sense that the function

$$\bar{y}(x, \varepsilon) = \bar{y}_1(x, \varepsilon) C_1 + \bar{y}_2(x, \varepsilon) C_2, \quad (8)$$

where C_1 , C_2 are arbitrary constant elements of Y , is the general solution of equation (3).

Proof. Substituting (5), (6) into (4), we obtain

$$2\bar{u}'_{10} + p(x)\bar{u}_{10} = \bar{\theta}, \dots, \quad 2\bar{u}'_{1i} + p(x)\bar{u}_{1i} = \bar{u}''_{1,i-1} + p(x)\bar{u}'_{1,i-1}, \dots; \quad (9)$$

$$2\bar{u}'_{20} + p(x)\bar{u}_{20} = \bar{\theta}, \dots, \quad 2\bar{u}'_{2i} + p(x)\bar{u}_{2i} + \bar{u}''_{2,i-1} + p(x)\bar{u}'_{2,i-1} = \bar{\theta}, \dots \quad (10)$$

Assume that the solutions $\bar{u}_{1i}(x)$, $\bar{u}_{2i}(x)$ ($i = 0, 1, 2, \dots$) of equations (9), (10) satisfy the initial conditions:

$$\text{for } x = x_0 \quad \bar{u}_{k0} = I, \quad \bar{u}_{ki} = \bar{\theta} \quad (k = 1, 2; i = 1, 2, \dots). \quad (11)$$

It is known (2) that there exist functions $\bar{y}_1^*(x, \varepsilon)$, $\bar{y}_2^*(x, \varepsilon)$ such that

$$\begin{aligned}\bar{y}_1^*(x, \varepsilon) &\sim \exp\left(-\frac{x-x_0}{\varepsilon}\right) \sum_{i=0}^{\infty} \bar{u}_{1i}(x) \varepsilon^i, \\ \bar{y}_2^*(x, \varepsilon) &\sim \exp\left(\frac{x-x_0}{\varepsilon}\right) \sum_{i=0}^{\infty} \bar{u}_{2i}(x) \varepsilon^i,\end{aligned}\tag{12}$$

and, moreover, $\bar{y}_1^*(x_0, \varepsilon) = \bar{y}_2^*(x_0, \varepsilon) = I$. The functions \bar{y}_1^* , \bar{y}_2^* are not, in general, solutions of equation (4), but with their aid we shall show the existence of the desired solutions $\bar{y}_1(x, \varepsilon)$, $\bar{y}_2(x, \varepsilon)$. Introduce

$$\bar{z}_k = \bar{y}_k - \bar{y}_k^*(x, \varepsilon) \quad (k = 1, 2),\tag{13}$$

where \bar{y}_k is a solution of equation (4) satisfying the conditions

$$\bar{y}_k(x_0, \varepsilon) = I, \quad \bar{y}_k'(x_0, \varepsilon) = \bar{y}_k^{*'}(x_0, \varepsilon).\tag{14}$$

Substituting $\bar{y}_k = \bar{z}_k + \bar{y}_k^*$ into (4), we obtain for \bar{z}_k the differential equation

$$\varepsilon^2 \bar{z}_k'' + \varepsilon^2 \bar{z}_k' = \bar{z}_k + \bar{a}_k(x, \varepsilon),\tag{15}$$

where

$$\bar{a}_k(x, \varepsilon) = \bar{y}_k^* - \varepsilon^2 \bar{y}_k^{*''} - \varepsilon^2 p(x) \bar{y}_k^{*'},\tag{16}$$

and, moreover, \bar{z}_k , on the basis of (13), (14), satisfies the initial conditions:

$$\text{for } x = x_0 \quad \bar{z}_k = \bar{\theta}, \quad \bar{z}_k' = \bar{\theta}.\tag{17}$$

Since $\bar{y}_k^*(x, \varepsilon)$ has the asymptotic expansion (12), which satisfies (4) formally, it follows from (16) that $\bar{a}_k(x, \varepsilon) \sim \bar{\theta}$, i.e.

$$\lim_{\varepsilon \rightarrow 0} \frac{\|\bar{a}_k(x, \varepsilon)\|}{\varepsilon^{N+2m}} = 0$$

for all positive numbers N , m , or

$$\|\bar{a}_k(x, \varepsilon)\| < \varepsilon^{N+2m}\tag{18}$$

for arbitrary positive numbers N , m and all sufficiently small $\varepsilon > 0$ ($\varepsilon \rightarrow 0$).

The solution \bar{z}_k of equation (15) under condition (17) will be written in the form

$$\bar{z}_k(x, \varepsilon) = \sum_{m=1}^{\infty} \bar{v}_{km}(x, \varepsilon), \quad (19)$$

where

$$\begin{aligned} \bar{v}_{k1}(x, \varepsilon) &= \frac{1}{\varepsilon^2} \int_{x_0}^x \int_{x_0}^{\tau_1} \bar{\alpha}_k(\tau_2, \varepsilon) d\tau_2 d\tau_1, \dots \\ \dots, \bar{v}_{k,m+1}(x, \varepsilon) &= \frac{1}{\varepsilon^2} \int_{x_0}^x \int_{x_0}^{\tau_1} (\bar{v}_{km}(\tau_2, \varepsilon) - \varepsilon^2 p(\tau_2) \bar{v}'_{km}(\tau_2, \varepsilon)) d\tau_2 d\tau_1, \dots \end{aligned} \quad (20)$$

Hence, from (2), (18), we obtain

$$\|\bar{v}_{k,m+1}(x, \varepsilon)\| \leq (2K)^{m+1} \frac{(x - x_0)^{m+1}}{(m+1)!} \varepsilon^N. \quad (21)$$

It follows from (21) that

$$\bar{z}_k(x, \varepsilon) = \sum_{m=1}^{\infty} \bar{v}_{km}(x, \varepsilon) \rightarrow \bar{\theta} \quad \text{as } \varepsilon \rightarrow 0.$$

Similarly we obtain

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \bar{z}_k(x, \varepsilon) = \bar{\theta}, \dots, \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n} \bar{z}_k(x, \varepsilon) = \bar{\theta}, \dots,$$

i.e.,

$$\bar{z}_k(x, \varepsilon) \sim \bar{\theta}, \quad x_1 \leq x \leq x_2, \quad k = 1, 2. \quad (22)$$

Hence, from (12), (13), (5), (6) follow.

Let y_0, y'_0 be arbitrary prescribed elements of Y . For the solution $y = y(x, \varepsilon)$ of equation (3), from (7), (8) we have:

$$\text{for } x = x_0 \quad y = C_1 + C_2 = y_0, \quad y' = \bar{y}'_1(x_0, \varepsilon)C_1 + \bar{y}'_2(x_0, \varepsilon)C_2 = y'_0. \quad (23)$$

Hence

$$[\bar{y}'_2(x_0, \varepsilon) - \bar{y}'_1(x_0, \varepsilon)]C_1 = \bar{y}'_2(x_0, \varepsilon)y_0 - y'_0,$$

$$[\bar{y}'_2(x_0, \varepsilon) - \bar{y}'_1(x_0, \varepsilon)]C_2 = y'_0 - \bar{y}'_1(x_0, \varepsilon)y_0. \quad (24)$$

From (5), (6) we have

$$\bar{y}'_1(x_0, \varepsilon) = -\frac{1}{\varepsilon} + \bar{u}'_{10}(x_0) + \bar{\eta}_1(\varepsilon), \quad \bar{y}'_2(x_0, \varepsilon) = \frac{1}{\varepsilon} + \bar{u}'_{20}(x_0) + \bar{\eta}_2(\varepsilon),$$

where $\bar{\eta}_1(\varepsilon), \bar{\eta}_2(\varepsilon)$ tend to $\bar{\theta}$ as $\varepsilon \rightarrow 0$. Consequently, the operator $\bar{y}'_2(x_0, \varepsilon) - \bar{y}'_1(x_0, \varepsilon)$, for all sufficiently small $\varepsilon > 0$, has an inverse $[\bar{y}'_2(x_0, \varepsilon) - \bar{y}'_1(x_0, \varepsilon)]^{-1} \rightarrow \varepsilon \bar{Y}(\varepsilon)$, where $\bar{Y}(\varepsilon) \rightarrow \frac{1}{2}I$ as $\varepsilon \rightarrow 0$. Hence, from (24), we obtain $C_1 \rightarrow y_0/2$, $C_2 \rightarrow y_0/2$ as $\varepsilon \rightarrow 0$.

Theorem 2. The solution $y(x, \varepsilon)$ of equation (3), satisfying the initial conditions (23), tends in norm to infinity as $\varepsilon \rightarrow 0$ with order $\exp(1/\varepsilon)$. The derivative $y'(x, \varepsilon)$ behaves analogously.

Indeed, in (8) $\bar{y}_1(x, \varepsilon)C_1 \sim \theta$, while $\|\bar{y}_2(x, \varepsilon)C_2\|$ tends to ∞ as $\varepsilon \rightarrow 0$ with order $\exp(1/\varepsilon)$.

Example 1. Let Y be the space $C_{[a,b]}$ of continuous functions $y = v(t)$. Equation (1) takes the form

$$\mu \frac{\partial^2 y(x, t)}{\partial x^2} + \mu \int_a^b P(x, t; \tau) \frac{\partial y(x, \tau)}{\partial x} d\tau = y(x, t),$$

where the function $P(x, t; \tau)$ is continuous in all variables for $x_1 \leq x \leq x_2$, $a \leq t, \tau \leq b$, is infinitely differentiable with respect to x for $x_1 \leq x \leq x_2$; $\mu \rightarrow 0$ ($\mu > 0$).

Example 2. Let Y be an n -dimensional space. Equation (1) is transformed into the system

$$\mu y''_i + \mu \sum_{i=1}^n p_{ji}(x) y'_i = y_j \quad (j = 1, 2, \dots, n),$$

where the functions $p_{ji}(x)$ are continuous and infinitely differentiable for $x_1 \leq x \leq x_2$; $\mu \rightarrow 0$ ($\mu > 0$).

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CITED LITERATURE

1. L. A. Lyusternik, V. I. Sobolev, *Elements of Functional Analysis*, Moscow-Leningrad, 1951.
2. L. A. Groza, DAN, 135, No. 5 (1960).

Note: Figure translations are in progress. See original paper for figures.

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