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Abstract

Full Text

MATHEMATICS

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On the Problem of Factorization of Operators in Hilbert Space

(Presented by Academician V. I. Smirnov on 1 VI 1962)

1. As is known ⁽¹⁾, Gauss' s method for inverting a nonsingular matrix $A = \|a_{jk}\|_1^n$ shows that, under certain conditions, the inverse matrix A^{-1} can be represented, moreover uniquely, in the form of a product of three factors $A^{-1} = S_+DS_-$, where the middle one is a diagonal matrix, and the outer ones S_{\pm} are right and left triangular matrices with diagonal elements equal to one. Such a representation is possible if and only if all principal minors $D_r = |a_{jk}|_1^r \neq 0$ ($r = 1, 2, \dots, n$). In operator form, the indicated factors can be obtained by the formulas

$$S_+ = \sum_{j=1}^n \frac{D_j}{D_{j-1}} (P_{jAPj})^{(-1)} \Delta P_j, \quad S_- = \sum_{j=1}^n \frac{D_j}{D_{j-1}} \Delta P_j (P_{jAPj})^{(-1)},$$

$$D = \sum_{j=1}^n \frac{D_{j-1}}{D_j} \Delta P_j.$$

Let us explain the notation. With each matrix of order n there is associated an operator acting in the n -dimensional space E_n of complex vectors $\xi = \{\xi_j\}_1^n$ and denoted by the same letter as the matrix. By P_j ($j = 1, 2, \dots, n$) is denoted the projection operator in E_n which preserves the first j coordinates of the vector ξ and annihilates all the others; $\Delta P_j = P_j - P_{j-1}$ ($j = 1, 2, \dots, n$; $P_0 = 0$). Finally, $(P_{jAPj})^{(-1)}$ ($j = 1, 2, \dots, n$) denotes the operator acting in the subspace $P_{jE}n$ and there inverse to the operator P_{jAPj} (it corresponds to the matrix inverse to the truncated matrix $\|a_{lk}\|_1^j$).

In the case when A is a Hermitian matrix, necessarily $S_- = S_+^*$, and the representation $A^{-1} = S_+DS_+^*$ itself is easily obtained from the expansion of the form $(A\xi, \xi)$ into a sum of squares by the Lagrange-Jacobi method.

There is also known a continuous analogue of the indicated matrix factorization for a Fredholm integral operator with kernel $\delta(x - s) - K(x, s)$ ($a \leq x, s \leq b$), where $\delta(x)$ is the Dirac function and $K(x, s)$, for example, is a continuous kernel.

The role of the condition $D_j \neq 0$ ($j = 1, 2, \dots, n$) is now played by the condition that, for every c ($a < c \leq b$), the integral equation

$$\varphi(x) - \int_a^c K(x, s)\varphi(s) ds = f(x) \quad (a \leq x \leq c) \quad (1)$$

has the resolvent $\Gamma_c(x, s)$. We form the Volterra kernels $V_{\pm}(x, s)$, putting $V_-(x, s) = \Gamma_x(x, s)$ ($a \leq s \leq x \leq b$) and $V_-(x, s) = 0$ ($a \leq x < s \leq b$), and similarly $V_+(x, s) = \Gamma_s(x, s)$ ($a \leq x \leq s \leq b$) and $V_+(x, s) = 0$ ($a \leq s < x \leq b$). Then

$$\delta(x - s) + \Gamma_b(x, s) = (\delta(x - s) + V_+(x, s)) * (\delta(x - s) + V_-(x, s)), \quad (2)$$

where the symbol $*$ denotes the operation of composition of kernels defined in the square $a \leq x, s \leq b$. This factorization is a continuous analogue of the matrix factorization with three factors considered above; the diagonal factor in representation (2) is, naturally, absent, since the role of D_i is now played by the Fredholm determinant of equation (1), and therefore the ratios D_{i-1}/D_i correspond to unity. Equality (2) has been used for various purposes in papers (2-4). Below we shall give continuous analogues and generalizations of the indicated factorizations for various classes of operators in Hilbert space.

- Let \mathfrak{R} denote the ring of all bounded linear operators acting in a Hilbert space \mathfrak{H} . Let $\mathfrak{P} = \{P\}$ be an arbitrary maximal continuous chain of orthogonal projectors ⁽⁵⁾. The chain \mathfrak{P} is called a **proper** chain of an operator $A \in \mathfrak{R}$ if all subspaces $P\mathfrak{H}$ ($P \in \mathfrak{P}$) are invariant subspaces of the operator A , i.e. $PAP = AP$ ($P \in \mathfrak{P}$). By \mathfrak{P}^{\perp} we denote the chain consisting of all projectors $I - P$ ($P \in \mathfrak{P}$). It is obvious that the sets $\mathfrak{R}^+(\mathfrak{P})$ and $\mathfrak{R}^-(\mathfrak{P}) = \mathfrak{R}^+(\mathfrak{P}^{\perp})$ of all operators possessing, respectively, the chains \mathfrak{P} and \mathfrak{P}^{\perp} , are closed subrings of the ring \mathfrak{R} .

In accordance with ⁽⁵⁾, by \mathfrak{S}_{∞} we shall denote the ideal of all completely continuous operators from \mathfrak{R} , and by $\mathfrak{S}_{\infty}^+(\mathfrak{P})$ and $\mathfrak{S}_{\infty}^-(\mathfrak{P})$ the ideals of the subrings $\mathfrak{R}^+(\mathfrak{P})$ and $\mathfrak{R}^-(\mathfrak{P})$ that are their intersection with \mathfrak{S}_{∞} .

By a **factorization** of an operator $A \in \mathfrak{R}$ with respect to a chain \mathfrak{P} we shall mean a representation of the operator A in the form of a product: $A = A_+A_-$, where $A_{\pm} \in \mathfrak{R}^{\pm}(\mathfrak{P})$. If a factorization for a given A ($\in \mathfrak{R}$) is possible, then it is not unique. Indeed, from a given factorization one can obtain an infinite set of others by putting, for example, $A'_+ = A_+B$ and $A'_- = B^{-1}A_-$, where B is an arbitrary invertible operator commuting with all projectors $P \in \mathfrak{P}$. However, if a factorization is possible which in what follows we shall call **special**:

$$A = (I + X_+)(I + X_-) \quad (X_{\pm} \in \mathfrak{S}_{\infty}^{\pm}(\mathfrak{P})), \quad (3)$$

then it will already be unique. Indeed, in this case X_{\pm} will be Volterra operators (^{5,6}), and consequently the operators $I + X_{\pm}$ (and together with them the operator A) will be invertible. If, moreover, one takes into account that the intersection of $\mathfrak{S}_{\infty}^{+}(\mathfrak{P})$ and $\mathfrak{S}_{\infty}^{-}(\mathfrak{P})$ consists only of the zero operator, then from this it is already easy to conclude the uniqueness of the factorization (3). From this uniqueness it follows that if a self-adjoint operator A ($A = A^*$) admits the special factorization (3), then in it $X_{-} = X_{+}^*$. Therefore such an operator A is always positive. We also note that, when seeking a formula for the operators X_{\pm} , without loss of generality we may immediately assume that the operator A has the form $A = (I - T)^{-1}$, where $T \in \mathfrak{S}_{\infty}$.

3. Let an arbitrary operator-function $F(P)$ be given, defined on the chain \mathfrak{P} and taking its values as operators from \mathfrak{R} . With its aid one can form a function $S(\mathfrak{z})$ of partitions (multivalued), putting for each partition $\mathfrak{z} = \{0 = P_0 < P_1 < \dots < P_n = I; P_j \in \mathfrak{P}\}$ of the chain \mathfrak{P} :

$$S(\mathfrak{z}) = \sum_{j=1}^n F(Q_j)(P_j - P_{j-1}) \quad (P_{j-1} \leq Q_j \leq P_j; Q_j \in \mathfrak{P}). \quad (4)$$

If $S(\mathfrak{z})$ has as its limit (in the sense of S. Shatunovskii) in the norm of operators some operator Y ($\in \mathfrak{R}$), i.e. if for every $\varepsilon > 0$ there exists a partition $\mathfrak{z}_{\varepsilon}$ of the chain \mathfrak{P} such that for all partitions \mathfrak{z} of this chain containing $\mathfrak{z}_{\varepsilon}$, every value of the function $S(\mathfrak{z})$ satisfies the condition

$\|S(\mathfrak{z}) - Y\| < \varepsilon$, then we shall write

$$Y = \int_{\mathfrak{P}} F(P) dP \quad (5)$$

and say that the integral appearing on the right-hand side of (5) converges (^{5,6}).

Along with the integral (5) we shall also need the integral

$$Y = \int_{\mathfrak{P}}^{+} F(P) dP,$$

whose definition differs from the definition of the integral (5) only in that, in the equality (4) defining the function $S(\delta)$, the projector Q_j is chosen in a completely definite way, namely $Q_j = P_j$ ($j = 1, 2, \dots, n$).

Theorem 1. *In order that the operator $A = (I - T)^{-1}$ ($A \in \mathfrak{R}$, $T \in \mathfrak{S}_{\infty}$) admit a special factorization with respect to a continuous maximal chain \mathfrak{P} , it is necessary and sufficient that: a) all operators $I - PTP$ ($P \in \mathfrak{P}$) be invertible, and b) at least one of the integrals*

$$X_+ = \int_{\mathfrak{P}} (I - PTP)^{-1} PT dP, \quad X_- = \int_{\mathfrak{P}} dPTP(I - PTP)^{-1} \quad (6)$$

converge.

If at least one of these integrals converges, then the other also converges, and their values give the special factorization (3).

Let us also note that in the formulation of Theorem 1 the integrals (6) may be replaced by the following:

$$X_+ = \int_{\mathfrak{P}}^+ [(I - PTP)^{-1} - I] dP, \quad X_- = \int_{\mathfrak{P}}^+ dP [(I - PTP)^{-1} - I].$$

4. A **normed ideal** (\mathfrak{I}) of the ring \mathfrak{R} is any ideal \mathfrak{S} of the ring \mathfrak{R} in which its own norm $|X|_{\mathfrak{S}}$ ($X \in \mathfrak{S}$) is defined, turning it into a Banach space and such that

$$|XAY|_{\mathfrak{S}} \leq \|X\| |A|_{\mathfrak{S}} \|Y\| \quad (X, Y \in \mathfrak{R}, A \in \mathfrak{S}).$$

Theorem 2. Let \mathfrak{P} be some continuous maximal chain, and let \mathfrak{S}_I and \mathfrak{S}_{II} be two normed ideals of the ring \mathfrak{R} .

If the following conditions are satisfied: 1) $\mathfrak{S}_I \subset \mathfrak{S}_{II}$; 2) the finite-dimensional operators form a dense set in \mathfrak{S}_I ; and 3) for every $T \in \mathfrak{S}_I$ the integral

$$S = \int_{\mathfrak{P}} PT dP$$

converges and belongs to \mathfrak{S}_{II} , then for every $T \in \mathfrak{S}_I$ satisfying the necessary condition a) (from Theorem 1), the integrals (6) converge, the operators $X_{\pm} \in \mathfrak{S}_{II}$, and, consequently, the operator $A = (I - T)^{-1}$ admits the special factorization (3) with operators $X_{\pm} \in \mathfrak{S}_{II}$.

It is not difficult to see that if an operator $T \in \mathfrak{S}_I$ satisfies the necessary condition a), then all operators in some neighborhood of it in \mathfrak{S}_I will also satisfy this condition, and the integrals (6) will map this neighborhood continuously into \mathfrak{S}_{II} .

By virtue of the results ^(5,8-11), it follows from Theorem 2 that as \mathfrak{S}_I one may, for example, take the normed ideals \mathfrak{S}_1 , \mathfrak{S}_p ($1 < p < \infty$), \mathfrak{S}_{ω} , and then as \mathfrak{S}_{II} , respectively, the ideals \mathfrak{S}_{Ω} , \mathfrak{S}_p , \mathfrak{S}_{∞} .

From Theorem 2 and what has been said we obtain:

Corollary. *If the operator $I-H$ ($H \in \mathfrak{S}_\omega$) is positive ($(Hf, f) < (f, f)$, $f \neq 0$), then it admits a special factorization with respect to any continuous maximal chain \mathfrak{F} .*

Apparently, this proposition is “sharp” in the sense that if H does not belong to \mathfrak{S}_ω , then there will be found a chain \mathfrak{F} with respect to which the special factorization of the operator $I-H$ proves impossible.

5. In the proof of Theorem 2 an essential role is played by the following general

Lemma. Let R be an arbitrary normed ring with identity e ($|e|_R = 1$), containing two closed subrings R_\pm which intersect only in zero. Suppose, moreover, that in R there exists a right (left) ideal J with the following properties: a) J is provided with its own norm $|c|_J$ ($c \in J$), making J a Banach space; b) there exists a constant $k(> 0)$ such that

$$|cx|_J \leq k|c|_J|x|_R \quad (|xc|_J \leq k|c|_J|x|_R) \quad (x \in R, c \in J);$$

c) every element $c \in J$ can be represented uniquely in the form $c = c_+ + c_-$, where $c_\pm \in R_\pm$, and, moreover,*

$$\sup_{c \in J} (|c_+|_R / |c|_J) < \infty.$$

Then there exists a number δ ($0 < \delta < 1$) such that, for every $a \in J$ with norm $|a|_J < \delta$, the element $(e - a)^{-1}$ can be represented in the form

$$(e - a)^{-1} = (e + b_+)(e + b_-) \quad (b_\pm \in R_\pm), \quad (7)$$

and if J is a right ideal, then

$$b_+ = a_+ + (aa_+)_+ + (a(aa_+)_+)_+ + \dots, \quad (8)$$

while if J is a left ideal, then

$$b_- = a_- + (a_-a)_- + ((a_-a)_-a)_- + \dots. \quad (9)$$

The convergence of the series (8) and (9) is understood in the sense of convergence in the norm of the ring R .

In proving Theorem 2 the lemma is used for the case when J is a two-sided ideal in R and when, consequently, the decompositions (8) and (9) hold simultaneously. Propositions akin to the lemma in content and proof are found in papers (12,13).

In the proof it turns out that the number δ can be chosen so small that the elements $e + b_\pm$ are invertible and $(e + b_\pm)^{-1} - e \in R_\pm$.

Let the set D consist of all elements $a \in J$ for which the element $e - a$ is invertible and $(e - a)^{-1}$ admits a factorization (7) with invertible factors $e + b_\pm$,

for which $(e + b_{\pm})^{-1} - e \in R_{\pm}$. Such a factorization for $a \in D$ will be unique. From the lemma one can derive that, under the conditions specified in it, D is an open part of J and is mapped continuously into R_{\pm} by the correspondence operators $a \rightarrow b_{+}$ and $a \rightarrow b_{-}$.

6. The results presented are generalized to the case of factorization of operators with three factors relative to an arbitrary discontinuous maximal chain \mathfrak{P} . They make it possible to obtain triangular representations for operators “close” to unitary ones, which will be set forth elsewhere.

The factorizations used in the theory of scalar and vector Wiener–Hopf integral equations and their discrete analogues^(14,15) are also examples of factorization in the sense of the general definition given in § 1, although the operators considered in these theories are bounded operators of a different type than in Theorems 1 and 2.

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REFERENCES

1. D. K. Faddeev, V. N. Faddeeva, *Computational Methods of Linear Algebra*, 1960.
2. M. G. Krein, DAN, 100, No. 3 (1955).
3. N. P. Sergeev, Zap. Leningrad. Gorn. Inst., 33, 3 (1956).
4. S. L. Sobolev, Izv. AN SSSR, ser. matem., 20, No. 4 (1956).
5. I. Ts. Gokhberg, M. G. Krein, DAN, 137, No. 5 (1961).
6. M. S. Brodskii, UMN, 16, 1 (1961).
7. R. Schatten, *A Theory of Cross-Spaces*, Princeton, 1950.
8. I. Ts. Gokhberg, M. G. Krein, DAN, 128, No. 2 (1959).
9. V. I. Matsaev, DAN, 139, No. 3 (1961).
10. V. I. Matsaev, DAN, 139, No. 4 (1961).

11. I. Ts. Gokhberg, M. G. Krein, DAN, 139, No. 4 (1961).
12. G. Baxter, Pacific J. Math., 8, 649 (1958).
13. N. Wiener, P. Masani, Acta Math., 99, 93 (1958).
14. M. G. Krein, UMN, 13, 5 (1958).
15. I. Ts. Gokhberg, M. G. Krein, UMN, 13, 2 (1958).

* This condition will certainly be satisfied if $\sup_{c \in J} (|c|_R / |c|_J) < \infty$.

Note: Figure translations are in progress. See original paper for figures.

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