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Abstract

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ON THE STATISTICAL THEORY OF HOMOGENEOUS ISOTROPIC TURBULENCE IN A RELATIVISTIC PLASMA

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To characterize the microscopic state of a relativistic plasma one may use the density functions in phase space

$$N_a(\mathbf{q},\mathbf{p},t) = \sum_{1<i<N_a} \delta(\mathbf{q}-\mathbf{q}_i(t))\,\delta(\mathbf{p}-\mathbf{p}_i(t))$$

and the strengths of the microscopic fields $\mathbf{E}^{(\mathrm{m})}, \mathbf{H}^{(\mathrm{m})}$; here N_a is the total number of particles of species a. The equations for the functions $N_a, \mathbf{E}^{(\mathrm{m})}, \mathbf{H}^{(\mathrm{m})}$ have the form (^{1})

$$\frac{\partial N_a}{\partial t} + \mathbf{v}\frac{\partial N_a}{\partial \mathbf{q}} + e_a\left(\mathbf{E}^{(\mathrm{m})} + \frac{1}{c}[\mathbf{v}\mathbf{H}^{(\mathrm{m})}]\right)\frac{\partial N_a}{\partial \mathbf{p}} = 0, \tag{1}$$

$$\mathbf{p} = m_a\mathbf{v}/\sqrt{1-v^2/c^2};$$

$$\operatorname{rot}\mathbf{H}^{(\mathrm{m})} = -\frac{1}{c}\frac{\partial \mathbf{E}^{(\mathrm{m})}}{\partial t} + 4\pi\sum_a e_a\int \mathbf{v}N_a\,d\mathbf{p}, \qquad \operatorname{div}\mathbf{H}^{(\mathrm{m})} = 0; \tag{2}$$

$$\operatorname{rot}\mathbf{E}^{(\mathrm{m})} = -\frac{1}{c}\frac{\partial \mathbf{H}^{(\mathrm{m})}}{\partial t}, \qquad \operatorname{div}\mathbf{E}^{(\mathrm{m})} = 4\pi\sum_a e_a\int N_a\,d\mathbf{p}. \tag{3}$$

Considering the functions $N_a, \mathbf{E}^{(\mathrm{m})}, \mathbf{H}^{(\mathrm{m})}$ as random, we shall confine ourselves here to an approximation in which the statistical properties are determined by specifying the first and second moments of these random functions. Such an approximation is, in particular, justified if the parameter $\varepsilon = 1/nr^3d \ll 1$.

In a spatially homogeneous plasma $\overline{N}_a = n_a f_a$, $\overline{\mathbf{E}^{(\mathrm{m})}} = 0$, $\overline{\mathbf{H}^{(\mathrm{m})}} = 0$, while the second moments depend only on the coordinate difference $\mathbf{q}-\mathbf{q}'$. Here $f_a = f_a(\mathbf{p},t)$ are the first distribution functions.

Introduce the functions

$$\delta N_a = N_a - \overline{N}_a, \qquad \delta\mathbf{E} = \mathbf{E}^{(\mathrm{m})} - \overline{\mathbf{E}^{(\mathrm{m})}} = \mathbf{E}^{(\mathrm{m})}, \qquad \delta\mathbf{H} = \mathbf{H}^{(\mathrm{m})} - \overline{\mathbf{H}^{(\mathrm{m})}} = \mathbf{H}^{(\mathrm{m})}. \tag{4}$$

In the second-moment approximation, in the equations for $\delta N_a, \delta\mathbf{E}, \delta\mathbf{H}$ only the linear terms may be retained. As a result, from equations (1)–(3) it follows that the equations for the functions $f_a, \delta N_a, \delta\mathbf{E}, \delta\mathbf{H}$ in a homogeneous isotropic plasma have the form

$$\frac{\partial f_a}{\partial t} = -\frac{e_a}{n_a}\frac{\partial}{\partial p_i}(\delta N_a \delta E_i) = -\frac{1}{8\pi^3}\frac{e_a}{n_a}\frac{\partial}{\partial p_i}\int(\delta N_a\delta E_i)_k\, d\mathbf{k}; \tag{5}$$

$$\frac{\partial \delta N_a}{\partial t} + \mathbf{v}\frac{\partial \delta N_a}{\partial \mathbf{q}} + e_a n_a \delta\mathbf{E}\,\frac{\partial f_a}{\partial \mathbf{p}} = 0; \tag{6}$$

$$\operatorname{rot}\delta\mathbf{H} = \frac{1}{c}\frac{\partial \delta\mathbf{E}}{\partial t} + 4\pi\sum_a e_a\int \mathbf{v}\,\delta N_a\, d\mathbf{p}, \qquad \operatorname{div}\delta\mathbf{H} = 0; \tag{7}$$

$$\operatorname{rot}\delta\mathbf{E} = -\frac{1}{c}\frac{\partial\delta\mathbf{H}}{\partial t}, \qquad \operatorname{div}\delta\mathbf{E} = 4\pi\sum_a e_a\int\delta N_a\, d\mathbf{p}. \tag{8}$$

In equation (5)

$$(\delta N_a\delta E_i)_k = \int \overline{\delta N_a \delta E_i}\, e^{-i\mathbf{k}(\mathbf{q}-\mathbf{q}')}\, d(\mathbf{q}-\mathbf{q}').$$

We introduce analogous notation also for other spatial spectral functions, for example $(\delta\mathbf{E}\delta\mathbf{E})_k$.

As in the nonrelativistic approximation (2), the spectral functions $(\delta N_a \delta N_b)_{\omega,k}$, $(\delta N_a\delta N_b)_k$, $(\delta N_a \delta E_i)_{\omega,k}$, $(\delta E_i \delta E_i)_{\omega,k}$, etc., cannot be expressed in terms of the one-particle distribution function f_a for the entire spectral range. For the part of the spectrum where the plasma is transparent, the spectral functions are expressed in terms of the functions $(\delta\mathbf{E}^{(\|)}\delta\mathbf{E}^{(\|)})_k^{(\mathrm{i})}$, $(\delta\mathbf{E}^{(\perp)}\delta\mathbf{E}^{(\perp)})_k^{(\mathrm{i})}$:

$$(\delta\mathbf{E}\delta\mathbf{E})_k^{(\mathrm{i})} = (\delta\mathbf{E}^{(\|)}\delta\mathbf{E}^{(\|)})_k^{(\mathrm{i})} + (\delta\mathbf{E}^{(\perp)}\delta\mathbf{E}^{(\perp)})_k^{(\mathrm{i})}, \tag{9}$$

the spatial spectral functions of the electric field in the transparency region. The index (i) indicates that in this region the radiation of plasma waves is essential. In a homogeneous plasma, the transverse ($\perp$) and longitudinal ($\|$) components of the vector $\delta\mathbf{E}$ do not correlate with one another. This follows directly from equations (6)–(8), and also from the theorems proved by A. M. Obukhov (3).

From equations (5)–(8) we obtain a closed system of equations for the functions f_a, $(\delta\mathbf{E}^{(\|)}\delta\mathbf{E}^{(\|)})^{(\mathrm{i})}_k$, $(\delta\mathbf{E}^{(\perp)}\delta\mathbf{E}^{(\perp)})^{(\mathrm{i})}_k$:

$$\frac{\partial f_a}{\partial t} = \frac{\partial}{\partial p_i}\left(D_{ij}^{a(\mathrm{st})} + D_{ij}^{a(\mathrm{i})}\right)\frac{\partial f_a}{\partial p_j} + \frac{\partial}{\partial p_i}\left[(A_i^{a(\mathrm{st})} + A_i^{a(\mathrm{i})})f_a\right], \quad i,j = 1,2,3; \tag{10}$$

$$\frac{\partial(\delta\mathbf{E}^{(\|)}\delta\mathbf{E}^{(\|)})^{\mathrm{i}}_k}{\partial t} =$$

$$= \pi\sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\int\left\{\mathbf{k}\frac{\partial f_a}{\partial\mathbf{p}}\frac{(\delta\mathbf{E}^{(\|)}\delta\mathbf{E}^{(\|)})^{(\mathrm{i})}_k}{4\pi} + (\mathbf{kv})f_a\right\}B^{(\|)}(\mathbf{kv},\mathbf{k})\,d\mathbf{p} =$$

$$= 8\pi P_k^{(\|)}; \tag{11}$$

$$\frac{\partial\{(\delta\mathbf{E}^{(\perp)}\delta\mathbf{E}^{(\perp)})^{\mathrm{i}}_k + (\delta\mathbf{H}\delta\mathbf{H})^{(\mathrm{i})}_k\}}{\partial t} =$$

$$= \pi\sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\int\left\{[[\mathbf{kv}]\,\mathbf{k}]_i\frac{\partial f_a}{\partial p_i}\frac{(\delta\mathbf{E}^{(\perp)}\delta\mathbf{E}^{(\perp)})^{\mathrm{i}}_k}{8\pi} + [\mathbf{kv}]^2 f_a\right\}\times$$

$$\times(\mathbf{kv})B^{(\perp)}(\mathbf{kv},\mathbf{k})\,d\mathbf{p} = 8\pi P_k^{(\perp)}. \tag{12}$$

The coefficients entering these equations are determined by the expressions

$$D_{ij}^{a(\mathrm{st})} = \sum_b 2e_a^2 e_b^2 n_b\int\left[\frac{k_i k_j/k^4}{|\varepsilon^{(\|)}(\mathbf{kv},\mathbf{k})|^2} + \frac{(\delta_{ij}k^2 - k_i k_j)(\mathbf{kv})^2[\mathbf{kv}']^2}{2k^4|(\mathbf{kv})^2\varepsilon^{(\perp)}(\mathbf{kv},\mathbf{k}) - c^2k^2|^2}\right]\times$$

$$\times\delta(\mathbf{kv} - \mathbf{kv}')f_b(\mathbf{p}')\,d\mathbf{k}\,d\mathbf{p}'; \tag{13}$$

$$A_i^{a(\mathrm{st})} = -\sum_b 2e_a^2 e_b^2 n_b\int\left[\frac{k_i k_j/k^4}{|\varepsilon^{(\|)}(\mathbf{kv},\mathbf{k})|^2} + \frac{[[\mathbf{kv}]\,\mathbf{k}]_i(\mathbf{kv})^2[[\mathbf{kv}']\,\mathbf{k}]_j}{2k^4|(\mathbf{kv})^2\varepsilon^{(\perp)}(\mathbf{kv},\mathbf{k}) - c^2k^2|^2}\right]\times$$

$$\times\delta(\mathbf{kv} - \mathbf{kv}')\frac{\partial f_b}{\partial p_j'}\,d\mathbf{k}\,d\mathbf{p}'; \tag{14}$$

$$D_{ij}^{a(\mathrm{i})} = \frac{e_a^2}{2\pi}(\mathbf{kv})\left[\frac{B^{(\|)}(\mathbf{kv},\mathbf{k})}{(\mathbf{kv})^2}\frac{(\delta E_i^{(\|)}\delta E_j^{(\|)})_k^{\mathrm{i}}}{4\pi} + B^{(\perp)}(\mathbf{kv},\mathbf{k})\frac{(\delta E_i^{(\perp)}\delta E_j^{(\perp)})_k^{\mathrm{i}}}{4\pi}\right] +$$

$$+\frac{e_a^2}{2\pi^2\omega_L^2}\int\left[\frac{k_i k_j}{k^2}P_k^{(\|)} + \frac{1}{z}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)P_k^{(\perp)}\right]d\mathbf{k}; \qquad (15)$$

$$A_i^{a(\mathrm{i})} = \frac{e_a^2}{2\pi}\left[\frac{k_i}{k^2}B^{(\|)}(\mathbf{kv},\mathbf{k}) + \frac{(\mathbf{kv})[[\mathbf{kv}]\,\mathbf{k}]_i}{k^2}B^{(\perp)}(\mathbf{kv},\mathbf{k})\right]; \qquad (16)$$

$$B^{(\|)}(\omega,\mathbf{k}) = \mathrm{sign}\left[\frac{\partial}{\partial\omega}\varepsilon'^{(\|)}(\omega,k)\right]\delta\big(\varepsilon'^{(\|)}(\omega,k)\big)\,; \qquad (17)$$

$$B^{(\perp)}(\omega,\mathbf{k}) = \mathrm{sign}\left[\frac{\partial}{\partial\omega}\left(\omega^2\varepsilon'^{(\perp)}(\omega,k)\right)\right]\delta\big(\omega^2\varepsilon'^{(\perp)}(\omega,k) - c^2k^2\big)\,; \qquad (18)$$

here $\varepsilon^{(\|)}(\omega,\mathbf{k}) = \varepsilon'^{(\|)} + \varepsilon''^{(\|)}$ is the longitudinal, and $\varepsilon^{(\perp)}(\omega,\mathbf{k}) = \varepsilon'^{(\perp)} + \varepsilon''^{(\perp)}$ the transverse dielectric permittivity of the plasma. In (15)

$$(\delta E_i^{(\|)}\delta E_j^{(\|)})_k^{(\mathrm{i})} = \frac{k_i k_j}{k^2}(\delta E^{(\|)}\delta E^{(\|)})_k^{(\mathrm{i})}, \qquad (\delta E_i^{(\perp)}\delta E_j^{(\perp)})_k^{(\mathrm{i})} =$$

$$= \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)(\delta E^{(\perp)}\delta E^{(\perp)})_k^{(\mathrm{i})}, \qquad (\delta H\delta H)_k^{(\mathrm{i})} = \frac{c^2k^2}{\omega_L^2 + c^2k^2}(\delta E^{(\perp)}\delta E^{(\perp)})_k^{\mathrm{i}}. \qquad (19)$$

The coefficients $D_{ij}^{a(\mathrm{st})}$, $A_i^{a(\mathrm{st})}$ refer to the region of plasma opacity. The terms containing these coefficients describe, in equations (10), the “collisions”of charged particles in the plasma.

The spectral functions are also expressed in terms of the functions f_a, $(\delta E^{(\|)}\delta E^{(\|)})_k^{(\mathrm{i})}$, $(\delta E^{(\perp)}\delta E^{(\perp)})_k^{(\mathrm{i})}$. Thus, for example, the space-time spectral functions $(\delta E_i\delta E_j)_{\omega,k}$ are determined by the expression

$$(\delta E_i\delta E_j)_{\omega,k} = (\delta E_i^{(\|)}\delta E_j^{(\|)})_{\omega,k} + (\delta E_i^{(\perp)}\delta E_j^{(\perp)})_{\omega,k}, \qquad (20)$$

where

$$(\delta E^{(\|)}\delta E^{(\|)})_{\omega,k} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\int\frac{2\pi\delta(\omega - \mathbf{kv})f_a}{|\varepsilon^{(\|)}(\omega,\mathbf{k})|^2}\,d\mathbf{p}+$$

$$+2\pi\frac{B^{(\|)}(\omega,\mathbf{k})}{\omega}(\delta E^{(\|)}\delta E^{(\|)})_k^{(\mathrm{i})}; \tag{21}$$

$$(\delta E^{(\perp)}\delta E^{(\perp)})_{\omega,k}=\sum_a\frac{(4\pi)^2e_a^2n_a}{k^2}\omega^2\int\frac{2\pi\delta(\omega-\mathbf{kv})[\mathbf{kv}]^2f_a}{|\omega^2\varepsilon^{(\perp)}(\omega,\mathbf{k})-c^2k^2|^2}\,d\mathbf{p}+$$

$$+2\pi\omega B^{(\perp)}(\omega,\mathbf{k})(\delta E^{(\perp)}\delta E^{(\perp)})_k^{(\mathrm{i})} \tag{22}$$

are the spectral functions of the longitudinal and transverse components of the vector $\delta\mathbf{E}$.

For the region of opacity, the results obtained here coincide with the results of the author' s work with V. P. Silin (^{4}). For states close to equilibrium, equations (10) coincide with those obtained in the author' s work (^{1}). In the nonrelativistic case the equations obtained reduce to the equations of work (^{2}).

In the equilibrium case, equations (10)–(12) have the solution: f_a are Maxwell distributions,

$$(\delta E^{(\|)}\delta E^{(\|)})_k^{\mathrm{i}}=4\pi\varkappa T; \qquad (\delta E^{(\perp)}\delta E^{(\perp)})_k^{\mathrm{i}}=8\pi\varkappa T. \tag{23}$$

The expressions for the functions $(\delta E_i^{(\|)}\delta E_j^{(\|)})_{\omega,k}$, $(\delta E_i^{(\perp)}\delta E_j^{(\perp)})_{\omega,k}$ in the equilibrium case coincide with those given in the book (^{5}).

If the plasma is in a vacuum, then the right-hand side of equation (12) vanishes, since $\omega^2\varepsilon^{(\perp)}(\omega,\mathbf{k})-c^2k^2\neq 0$ for $\omega=\mathbf{kv}$. This means that, in the region of transparency, transverse waves do not interact with the charged particles of the plasma. However, such an interaction takes place if the plasma is in some retarding system. Then plasma transverse waves are possible whose propagation velocity is less than the speed of light in vacuum. Transverse waves also interact with charged particles in the presence of an external magnetic field.

Starting from equations (5)–(8), let us write the expressions of the conservation laws for the case under consideration of a homogeneous and isotropic plasma. The conservation laws for the total particle-number density and the momentum density have the form:

$$\sum_a n_a=\mathrm{const}, \qquad \sum_a n_a\int\mathbf{p}f_a\,d\mathbf{p}=0. \tag{24}$$

In deriving the momentum conservation law it has been taken into account that, in a homogeneous isotropic plasma, $\overline{\delta\rho\,\delta\mathbf{E}}=0$, i.e. the correlation of a vector and a scalar is equal to zero.

Taking into account that in the case under consideration the correlation of the electric- and magnetic-field intensities is equal to zero, we can write the law of energy conservation in the form

$$\frac{\partial}{\partial t}\left\{\sum_a n_a \int \frac{p^2}{2m_a} f_a \, dp + \frac{1}{8\pi}\left[\delta E \, \delta E + \left(\overline{\delta H \delta H}\right)_k\right]\right\} = 0. \tag{25}$$

If one takes into account that

$$\overline{(\delta E \, \delta E)} = \frac{1}{(2\pi)^4} \int \left[(\delta E^{(\|)} \delta E^{(\|)})_{\omega,k} + (\delta E^{(\perp)} \delta E^{(\perp)})_{\omega,k}\right] d\omega \, dk,$$

then from formulas (20)–(22) it follows that the energy in a homogeneous and isotropic plasma can be expressed in terms of the functions f_a, $(\delta E^{(\|)} \delta E^{(\|)})_k^{(\mathrm{II})}$, $(\delta E^{(\perp)} \delta E^{(\perp)})_k^{(\mathrm{II})}$.

It should be emphasized that the equations given here are valid when it is possible to restrict oneself to taking into account only two roots of each of the equations $\varepsilon'^{(\|)}(\omega_k, k) = 0$, $\varepsilon'^{(\perp)}(\omega_k, k) - c^2 k^2 = 0$. Otherwise the number of equations increases.

Above it was assumed that $\omega_k^2 \sim \omega_L^2$, but writing the formulas for the more general case presents no difficulties.

A detailed derivation of the equations presented here is given in the author' s dissertation[6].

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Note: Figure translations are in progress. See original paper for figures.