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Abstract

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ON THE THEORY OF A GYROSCOPIC TRACKING SYSTEM IN THE PRESENCE OF RANDOM DISTURBANCES

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A gyroscopic tracking system is a gyroscope with three degrees of freedom, operating on the principle of force gyroscopic stabilization. The axis of the outer gimbal ring of the gyroscope is vertical, and the axis of the gyroscope casing is horizontal. The outer gimbal ring must track a useful input signal specified externally by some command device. The input signal consists of the useful signal and a disturbance, which are stationary random processes whose correlation functions are assumed known.

In the present work, on the basis of the criterion of minimum mean-square error [1–3], the problem of optimal reproduction of the useful signal by a gyroscopic tracking system is considered.

The equations of motion of the gyroscopic tracking system have the form [4]

$$A\alpha'' + n_1\alpha' - H\beta' - m_1\beta = -m_1x_1(t), \qquad B\beta'' + H\alpha' + S\alpha = Sx_2(t). \quad (1)$$

Here α is the angle of rotation of the outer gimbal ring of the gyroscope; β is the angle of rotation of the gyroscope casing; H is the kinetic moment of the gyroscope; A is the moment of inertia of the gyroscope together with the casing and the outer gimbal ring with respect to the axis of this ring; B is the moment of inertia of the gyroscope together with the casing with respect to the casing axis; $-n_1\alpha'$ is the moment of friction forces in the supports of the axis of the outer gimbal ring of the gyroscope; $m_1[\beta - x_1(t)]$ is the moment about the axis of the outer gimbal ring of the gyroscope imposed by the stabilizing motor; $-S[\alpha - x_2(t)]$ is the moment about the axis of the gyroscope casing imposed by the correction electromagnet. The functions $x_1(t)$ and $x_2(t)$ will be determined below from the conditions of optimal reproduction of the useful signal.

The system of differential equations (1) can be replaced by the matrix differential equation

$$f(D)z(t) = e(D)x(t) \qquad (D = d/dt), \quad (2)$$

where

$$f(D)=\left\|\begin{matrix} D^2+\frac{n_1}{A}D & -\left(\frac{H}{A}D+\frac{m_1}{A}\right) \\ \frac{H}{B}D+\frac{S}{B} & D^2 \end{matrix}\right\|, \qquad z=\left\|\begin{matrix}\alpha \\ \beta\end{matrix}\right\|,$$

$$e(D)=\left\|\begin{matrix} -\frac{m_1}{A} & 0 \\ 0 & -\frac{S}{B} \end{matrix}\right\|, \qquad x(t)=\left\|\begin{matrix}x_1(t) \\ x_2(t)\end{matrix}\right\|. \tag{3}$$

From equation (2) it follows that

$$z(t)=Y(D)x(t) \qquad (Y(D)=F(D)e(D)/\Delta(D)). \tag{4}$$

Here $F(D)$ is the adjugate matrix of the matrix $f(D)$, and $\Delta(D)$ is the determinant of the matrix $f(D)$.

The matrix transfer function $Y(D)$, according to (4), has the form

$$Y(D)=\frac{1}{\Delta(D)}\left\|\begin{matrix} -\frac{m_1}{A}D^2 & \frac{S}{B}\left(\frac{H}{A}D+\frac{m_1}{A}\right) \\ \frac{m_1}{A}\left(\frac{H}{B}D+\frac{S}{B}\right) & \frac{S}{B}\left(D^2+\frac{n_1}{A}D\right) \end{matrix}\right\|, \tag{5}$$

$$\Delta(D)=D^4+\frac{n_1}{A}D^3+q^2D^2+\frac{S+m_1}{H}q^2D+\frac{Sm_1}{H^2}q^2 \qquad \left(q^2=\frac{H^2}{AB}\right).$$

The solution of the matrix differential equation (2) under zero initial conditions will be as follows:

$$z(t)=\int_0^t W(t-\tau)x(\tau)\,d\tau. \tag{6}$$

Here $W(t)$ is the weighting function of the system, which is the original corresponding to the image $pY(p) \div W(t)$, where $Y(p)$ is obtained from (5) by replacing the argument D by the Carson-Heaviside operator p. In the case where all roots x_σ of the characteristic equation $\Delta(p)=0$ are simple, the weighting function $W(t)$ can be represented in the form [5]

$$W(t)=\sum_\sigma \left[F(p)e(p)\frac{p-x_\sigma}{\Delta(p)}\right]_{p=x_\sigma} e^{x_\sigma t}.$$

The law of motion of the outer gimbal ring and the gyroscope case, according to (6), will be

$$\alpha(t) = \sum_{i=1}^{2} \int_0^t W_{1i}(t-\tau) x_i(\tau)\, d\tau, \qquad \beta(t) = \sum_{i=1}^{2} \int_0^t W_{2i}(t-\tau) x_i(\tau)\, d\tau, \quad (7)$$

where $W_{jk}(t)$ are the elements of the matrix weighting function $W(t)$.

As indicated above, the input signal entering the system is

$$\theta(t) = m(t) + n(t), \quad (8)$$

where $m(t)$ is the useful signal and $n(t)$ is the disturbance. The useful signal and the disturbance are mutually uncorrelated stationary random processes with mathematical expectations equal to zero. The correlation functions of these processes have the form

$$R_m(\tau) = Le^{-\mu|\tau|} \left(\cos \varepsilon\tau + \frac{\mu}{\varepsilon} \sin \varepsilon|\tau| \right), \qquad R_n(\tau) = Ne^{-\chi|\tau|}. \quad (9)$$

The spectral densities of the random processes $m(t)$ and $n(t)$, respectively, will be

$$S_m(\omega) = \frac{4\mu\nu^2 L}{(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2}, \qquad S_n(\omega) = \frac{2\chi N}{\omega^2 + \chi^2} \quad (\nu^2 = \varepsilon^2 + \mu^2). \quad (10)$$

The weighting function $\Gamma(t)$ of the optimal linear system reproducing the useful signal with minimum mean-square error is the solution of the integral equation [2]

$$R_{m\theta}(\tau) - \int_0^\infty R_\theta(\tau - \xi)\Gamma(\xi)\, d\xi = 0 \qquad (\tau \geqslant 0). \quad (11)$$

Here $R_\theta(\tau)$ is the correlation function of the random process $\theta(t)$, and $R_{m\theta}(\tau)$ is the cross-correlation function of the random processes $m(t)$ and $\theta(t)$.

The transfer function $\Phi(D)$ of the optimal system is related to the weighting function $\Gamma(t)$ by the operational relation $p\Phi(p) \div \Gamma(t)$, where p is the Carson-Heaviside operator. From the integral equation (11), we shall have for the optimal transfer function the expression

$$\Phi(i\omega) = \frac{1}{2\pi S_\theta^+(i\omega)} \int_0^\infty e^{-i\omega\tau}\, d\tau \int_{-\infty}^{\infty} \frac{S_{m\theta}(\omega)}{S_\theta^-(i\omega)} e^{i\omega\tau}\, d\omega. \quad (12)$$

Here $S_{m\theta}(\omega)$ is the cross-spectral density of the random processes $m(t)$ and $\theta(t)$. The functions $S_\theta^+(i\omega)$ and $S_\theta^-(i\omega)$ are complex conjugate functions obtained as a result of factorizing the spectral density $S_\theta(\omega)$, i.e., these functions satisfy the condition

$$S_\theta(\omega) = S_\theta^+(i\omega) S_\theta^-(i\omega),$$

where all zeros and poles of the function $S_\theta^+(i\omega)$ are situated in the upper half-plane, and all zeros and poles of the function $S_\theta^-(i\omega)$ in the lower half-plane of the complex variable ω.

In accordance with (10),

$$S_\theta(\omega) = \left| \sqrt{2\varkappa N} \, \frac{(i\omega)^2 + 2\mu_1(i\omega) + \nu_1^2}{[(i\omega)^2 + 2\mu(i\omega) + \nu^2](\varkappa + i\omega)} \right|^2, \tag{13}$$

where ν_1 and μ_1 are determined from the relations

$$\nu_1^4 = \nu^4 + \frac{4\mu\nu^2 L\varkappa}{N}, \qquad 4\mu_1^2 = 4\mu^2 + 2\nu_1^2 - 2\nu^2 + \frac{4\mu\nu^2 L}{2\varkappa N}. \tag{14}$$

From expression (13) we find that

$$S_\theta^+(\lambda) = \sqrt{2\varkappa N} \, \frac{\lambda^2 + 2\mu_1\lambda + \nu_1^2}{(\lambda^2 + 2\mu\lambda + \nu^2)(\varkappa + \lambda)}, \qquad S_\theta^-(\lambda) = \sqrt{2\varkappa N} \, \frac{\lambda^2 - 2\mu_1\lambda + \nu_1^2}{(\lambda^2 - 2\mu\lambda + \nu^2)(\varkappa - \lambda)},$$

$$\lambda = i\omega. \tag{15}$$

Since the random processes $m(t)$ and $n(t)$ are uncorrelated with one another, the cross-spectral density of these processes is $S_{m\theta}(\omega) = S_m(\omega)$. Therefore, in accordance with (10) and (15),

$$\frac{S_{m\theta}(\omega)}{S_\theta^-(i\omega)} = K(\lambda) = \frac{4\mu\nu^2 L}{\sqrt{2\varkappa N}} \, \frac{\varkappa - \lambda}{(\lambda^2 - 2\mu_1\lambda + \nu_1^2)(\lambda^2 + 2\mu\lambda + \nu^2)}. \tag{16}$$

In the case when $\varepsilon^2 = \nu^2 - \mu^2 > 0$, $\varepsilon_1^2 = \mu_1^2 - \nu_1^2 > 0$, the zeros of the denominator in expression (16) have the form

$$\lambda_1, \lambda_2 = \mu_1 \pm \varepsilon_1, \qquad \lambda_3, \lambda_4 = -\mu \pm \varepsilon i. \tag{17}$$

The function $K(\lambda)$ is fractional-rational, and it can be represented as the sum of two functions

$$K(\lambda) = K^+(\lambda) + K^-(\lambda), \tag{18}$$

where $K^+(\lambda)$ and $K^-(\lambda)$ are functions whose poles are situated, respectively, in the left and right half-planes of the complex variable λ.

Taking into account the form of the function $K(\lambda)$, it is not difficult to show that expression (12) for the optimal transfer function can be represented as

$$\Phi(i\omega) = \frac{K^+(i\omega)}{S_\theta^+(i\omega)}. \tag{19}$$

The function $K^+(\lambda)$, in accordance with (16) and (17), is reduced to the form

$$K^+(\lambda) = \frac{4\mu\nu^2 L}{\sqrt{2\varkappa N}} \frac{a}{[(\mu+\mu_1+\varepsilon_1)^2+\varepsilon^2][(\mu+\mu_1-\varepsilon_1)^2+\varepsilon^2]} \frac{\lambda+\rho}{\lambda^2+2\mu\lambda+\nu^2}, \tag{20}$$

$$\rho = b/a, \qquad a = \mu^2+\varepsilon^2+2\mu\varkappa+\varkappa(\lambda_1+\lambda_2)-\lambda_1\lambda_2,$$

$$b = 2\mu^3+(3\mu^2-\varepsilon^2)\varkappa+2\mu\varepsilon^2+(\mu^2+\varepsilon^2+2\mu\varkappa)(\lambda_1+\lambda_2)+\varkappa\lambda_1\lambda_2.$$

According to (19), (20), and (15), for the optimal transfer function we shall have the final expression

$$\Phi(D) = k\,\frac{(D+\varkappa)(D+\rho)}{(D+\lambda_1)(D+\lambda_2)}, \qquad k = \frac{2\mu\nu^2 L}{\varkappa N} \frac{a}{[(\mu+\mu_1+\varepsilon_1)^2+\varepsilon^2][(\mu+\mu_1-\varepsilon_1)^2+\varepsilon^2]}. \tag{21}$$

We now pass to the determination of the functions $x_1(t)$ and $x_2(t)$ entering into the differential equations (1). The best filtering of the input signal $\theta(t)$, from the point of view of the minimum mean-square error, can be achieved by passing this signal through a filter with transfer function $\Phi(D)$, determined by expression (21). The signal $y(t)$ at the output of this filter will be

$$y(t) = \int_0^t \Gamma(t-\tau)\theta(\tau)\,d\tau.$$

The optimal function-

of the gyroscopic tracking system will be attained when the conditions $\alpha(t) = y(t)$, $\beta(t) = 0$ are satisfied, i.e., when

$$\sum_{i=1}^{2} \int_0^t W_{1i}(t-\tau)x_i(\tau)\,d\tau = y(t), \qquad \sum_{i=1}^{2} \int_0^t W_{2i}(t-\tau)x_i(\tau)\,d\tau = 0. \tag{22}$$

From the system of integral equations (22) the signals $x_1(t)$ and $x_2(t)$, applied to the input of the gyroscope, are determined. Thus, the optimal gyroscopic tracking system will contain an optimal filter with transfer function $\Phi(D)$, to whose input the input signal $\theta(t)$ is supplied. The output signal of the optimal filter $y(t)$ is fed to the input of a computing device that solves the integral equations (22). The solutions of these equations, $x_1(t)$ and $x_2(t)$, are the signals that enter the input of the gyroscope.

As an example, let us consider the case of filtering the input signal $\theta(t) = m(t) + n(t)$, characterized by the spectral densities (10), whose parameters are: $L = 0.01$, $\mu = 0.2$ sec^{-1}, $\nu = 10$ sec^{-1}, $\varepsilon = 9.998$ sec^{-1}, $N = 16 \cdot 10^{-6}$, $\varkappa = 0.5$ sec^{-1}. For these data, $\nu_1 = 12.25$ sec^{-1}, $\mu_1 = 111.92$ sec^{-1}, $\varepsilon_1 = 111.24$ sec^{-1}, $\lambda_1 = 223$, $\lambda_2 = 0.68$, $a = 60.3$, $b = 22490$, $\rho = 373$, $k = 0.603$. The optimal transfer function, according to (21), will be $\Phi(D) = 0.603(D + 0.5)(D + 373)/(D + 0.68)(D + 223)$.

To evaluate the quality of filtering, let us find the variance of the error $E(t) = y(t) - m(t)$ in reproducing the useful signal. The error variance

$$\overline{E^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[|\Phi(i\omega) - 1|^2 S_m(\omega) + |\Phi(i\omega)|^2 S_n(\omega) \right] d\omega$$

will be

$$\overline{E^2} = 2\varkappa N k^2 I_3 + 4\mu\nu^2 L I_4,$$

$$I_3 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(i\omega)}{l(i\omega)l(-i\omega)} \, d\omega, \qquad I_4 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{g(i\omega)}{h(i\omega)h(-i\omega)} \, d\omega,$$

where $G(i\omega) = B_0(i\omega)^4 + B_1(i\omega)^2 + B_2$, $l(i\omega) = c_0(i\omega)^3 + c_1(i\omega)^2 + c_2(i\omega) + c_3$, $g(i\omega) = b_0(i\omega)^6 + b_1(i\omega)^4 + b_2(i\omega)^2 + b_3$,
$h(i\omega) = a_0(i\omega)^4 + a_1(i\omega)^3 + a_2(i\omega)^2 + a_3(i\omega) + a_4$, $B_0 = 1$, $B_1 = -(\varkappa^2 + \rho^2)$, $B_2 = \varkappa^2\rho^2$, $c_0 = 1$, $c_1 = 2\mu_1 + \varkappa$, $c_2 = 2\mu_1\varkappa + \nu_1^2$, $c_3 = \varkappa\nu_1^2$, $b_0 = 0$, $b_1 = (1-k)^2$, $b_2 = -\{2(1-k)(k\varkappa\rho - \nu_1^2) + [k(\varkappa + \rho) - 2\mu_1]^2\}$, $b_3 = (k\varkappa\rho - \nu_1^2)^2$, $a_0 = 1$, $a_1 = 2(\mu + \mu_1)$, $a_2 = \nu^2 + \nu_1^2 + 4\mu\mu_1$, $a_3 = 2(\mu\nu_1^2 + \mu_1\nu^2)$, $a_4 = \nu^2\nu_1^2$.

For the data given here, $I_3 = 2.14 \cdot 10^{-2}$, $I_4 = 2.05 \cdot 10^{-6}$. The variance and root-mean-square value of the error in reproducing the useful signal will be

$$\overline{E^2} = 1.76 \cdot 10^{-6}, \qquad \sqrt{\overline{E^2}} = 1.33 \cdot 10^{-3}.$$

Let us note that when the input signal $\theta(t)$ is fed directly to the gyroscope, i.e., when $x_1(t) \equiv 0$, $x_2(t) \equiv \theta(t)$, the reproduction error is very large. For example, for a gyroscope whose parameters are $S/H = 0.1$ sec^{-1}, $m_1/H = 1$ sec^{-1}, $q = 50$ sec^{-1}, $n_1/A = 10$ sec^{-1}, the root-mean-square error of reproduction of the useful signal considered here is $\sqrt{\overline{E^2}} = 68 \cdot 10^{-3}$, i.e., $\sqrt{\overline{E^2}} = 0.68\sqrt{\overline{m^2(t)}}$, which is, of course, completely unacceptable.

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Note: Figure translations are in progress. See original paper for figures.