CONJUGATE NETS WITH A PAIR \(T_1\) AS A PAIR OF CONGRUENCES OF AXES

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Abstract

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MATHEMATICS

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CONJUGATE NETS WITH A PAIR T_1 AS A PAIR OF CONGRUENCES OF AXES

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1. To each point A_0 of a conjugate net (A_0) we attach a projective frame formed by four analytic points: the point of the net A_0, the corresponding points A_1 and A_2 of the first Laplace transforms (A_1) and (A_2) of the net (A_0), and an arbitrary point A_3 lying on the first axis (1) of the net (A_0) at the point A_0. The infinitesimal projective displacements of the vertices of the frame are determined by the equations

$$dA_i = \omega_i^j A_j \qquad (i, j = 0, 1, 2, 3), \tag{1}$$

where ω_i^j are linear differential forms satisfying the structure equations of the projective space P_3:

$$D\omega_i^j = [\omega_i^k \omega_k^j] \qquad (i, j, k = 0, 1, 2, 3). \tag{2}$$

With the indicated choice of the vertices of the frame we shall have (2) the following Pfaffian equations:

$$\omega^3 = 0, \qquad \omega_1^2 = 0, \qquad \omega_2^1 = 0; \tag{3}$$

$$\omega_1^3 = a\omega^1, \qquad \omega_2^3 = c\omega^2. \tag{4}$$

Exterior differentiation of equations (3) with the aid of equations (2) and the application of Cartan' s lemma give

$$\begin{aligned} \omega_3^2 &= \alpha\omega^1 + \beta\omega^2, & \omega_2^0 &= \gamma_1\omega^1 - c\beta_1\omega^2, \\ \omega_1^0 &= -a\beta\omega^1 + \gamma\omega^2, & \omega_3^1 &= \beta_1\omega^1 + \alpha_1\omega^2. \end{aligned} \tag{5}$$

Exterior differentiation of equations (4) and (5) leads to the following quadratic equations:

$$
\begin{aligned}
&[\Delta a\,\omega^1]=0, \quad [\Delta\alpha\,\omega^1]+[\Delta\beta\,\omega^2]=0, \quad [\Delta\gamma_1\,\omega^1]-c[\Delta\beta_1\,\omega^2]=0,\\
&[\Delta c\,\omega^2]=0, \quad a[\Delta\beta\,\omega^1]-[\Delta\gamma\,\omega^2]=0, \quad [\Delta\beta_1\,\omega^1]+[\Delta\alpha_1\,\omega^2]=0,
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
\Delta a &= da + a(\omega_0^0 - 2\omega_1^1 + \omega_3^3), & \Delta c &= dc + c(\omega_0^0 - 2\omega_2^2 + \omega_3^3),\\
\Delta\alpha &= d\alpha + \alpha(\omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3), & \Delta\alpha_1 &= d\alpha_1 + \alpha_1(\omega_0^0 + \omega_1^1 - \omega_2^2 - \omega_3^3),\\
\Delta\beta &= d\beta + \beta(\omega_0^0 - \omega_3^3) - \omega_3^0, & \Delta\beta_1 &= d\beta_1 + \beta_1(\omega_0^0 - \omega_3^3) - \omega_3^0,\\
\Delta\gamma &= d\gamma + \gamma(2\omega_0^0 - \omega_1^1 - \omega_2^2), & \Delta\gamma_1 &= d\gamma_1 + \gamma_1(2\omega_0^0 - \omega_1^1 - \omega_2^2).
\end{aligned}
$$

2. The invariant differential forms corresponding to the point and tangential Darboux invariants of the net (A_0) are the forms [(2)]

$$
\begin{aligned}
H &= \gamma_1[\omega^1\omega^2], & \overline{H} &= c\alpha[\omega^1\omega^2],\\
K &= \gamma[\omega^1\omega^2], & \overline{K} &= a\alpha_1[\omega^1\omega^2].
\end{aligned} \tag{7}
$$

Denote by $F_{1,2} = \lambda_{1,2}A_0 + A_3$ the foci of the first axis, and by $\Phi_{1,2} = \mu_{1,2}A_0 + A_3$ the foci of the second. Then

$$
(dF, A_0, A_3) \equiv 0 \pmod{\omega^1 - t\omega^2},
$$

$$
(d\Phi, A_1, A_2) \equiv 0 \pmod{\omega^2 - \tau\omega^2},
$$

whence, using equations (1), (3), (4), (5), by virtue of the linear independence of the points A_i, we obtain that the numbers $\lambda_{1,2}$ and $\mu_{1,2}$ are the roots of the quadratic equations

$$
\lambda^2 + (\beta + \beta_1)\lambda + \beta\beta_1 - \alpha\alpha_1 = 0, \tag{8}
$$

$$
a\gamma\mu^2 + ac(\beta - \beta_1)\mu - c\gamma_1 = 0, \tag{9}
$$

and that the developable surfaces of the congruences (A_0A_3) and (A_1A_2) are determined, respectively, by the equations

$$
\alpha(\omega^1)^2 + (\beta - \beta_1)\omega^1\omega^2 - \alpha_1(\omega^2)^2 = 0, \tag{10}
$$

$$
a\gamma_1(\omega^1)^2 + ac(\beta - \beta_1)\omega^1\omega^2 - c\gamma(\omega^2)^2 = 0. \tag{11}
$$

Let the second axis A_1A_2 intersect the focal planes of the first axis A_0A_3 at the points $B_{1,2} = \bar{\lambda}_{1,2}A_0 + A_3$, and the first axis A_0A_3 intersect the focal planes of the second at the points $C_{1,2} = \bar{\mu}_{1,2}A_1 + A_2$. Then

$$(d\Phi, A_1, A_2, B) = 0, \qquad (dF, A_0, A_3, C) = 0,$$

whence we obtain that $\bar{\lambda}_{1,2}$ and $\bar{\mu}_{1,2}$ are the roots of the equations

$$ac\bar{\lambda}^2 + ac(\beta - \beta_1)\bar{\lambda} + ac\beta\beta_1 - \gamma\gamma_1 = 0, \tag{12}$$

$$\alpha\bar{\mu}^2 + (\beta - \beta_1)\bar{\mu} - \alpha_1 = 0. \tag{13}$$

3. R. M. Gaidel' man(3) introduced the concept of a pair of congruences T_1 (a "one-sided" pair $T^{(1)}$): this is such a pair of congruences, between whose rays a one-to-one correspondence is established, in which the focal planes of each ray of the first congruence pass through the foci of the corresponding ray of the second congruence.

Let us determine for which nets the pair of axis congruences forms a pair T_1 in the direction from the second axes to the first. For this, by definition, it is necessary that the foci $F_{1,2}$ coincide with the points $B_{1,2}$, i.e. that equations (8) and (12) coincide; and this will occur if and only if

$$\gamma\gamma_1 = ac\alpha\alpha_1. \tag{14}$$

By virtue of (7) this equality gives

$$HK = \overline{H}\,\overline{K}. \tag{15}$$

Differentiating (14), we obtain

$$\gamma\Delta\gamma_1 + \gamma_1\Delta\gamma = ac(\alpha_1\Delta\alpha + \alpha\Delta\alpha_1) + a\alpha_1(a\Delta c + c\Delta a). \tag{16}$$

The system of equations determining the nets (15) consists of the finite condition (14), the Pfaffian equations (3), (4), (5), (16), and the quadratic equations (6). The number of unknown forms of the characteristic system is $q = 7$, since one of the forms $\Delta a, \Delta c, \Delta\alpha, \Delta\beta, \Delta\gamma, \Delta\alpha_1, \Delta\beta_1, \Delta\gamma_1$ can be determined from (16). The number of independent quadratic equations is equal to 6; therefore $s_1 = 6$, $s_2 = q - s_1 = 1$, $Q = s_1 + 2s_2 = 8$. The most general solution of equations (6) has the form:

$$\Delta a = m\omega^1, \qquad \Delta c = n\omega^2,$$

$$\Delta\alpha = p\omega^1 + q\omega^2, \qquad \Delta\beta = q\omega^1 + r\omega^2, \qquad \Delta\gamma = -ar\omega^1 + s\omega^2, \tag{17}$$

$$\Delta\alpha_1 = q_1\omega^1 + p_1\omega^2, \qquad \Delta\beta_1 = r_1\omega^1 + q_1\omega^2, \qquad \Delta\gamma_1 = s_1\omega^1 - cr_1\omega^2.$$

In this case equation (16), by virtue of the linear independence of the forms ω^1, ω^2, leads to the equalities

$$\gamma s_1 - \gamma_1 ar = c(a\alpha q_1 + a\alpha_1 p + m\alpha\alpha_1),$$

$$\gamma_1 s - \gamma cr_1 = a(c\alpha_1 q + c\alpha p_1 + n\alpha\alpha_1). \tag{18}$$

Equations (17) and (18) show that the most general two-dimensional integral element depends on $N = 10 - 2 = 8$ arbitrary parameters. Since $N = Q$, the system is in involution and determines the nets (15) with the arbitrariness of one function of two arguments.

Theorem 1. *The pair of congruences of axes of a conjugate net forms a T_1 pair in the direction from the second axes to the first for those and only those nets for which the product of the Darboux point invariants is equal to the product of the tangential invariants. Such nets depend on one function of two arguments. Their characteristic property is the correspondence of the asymptotic lines on their first Laplace transforms.*

The last assertion follows from formulas (5) of paper [2].

4. It is known [4] that a one-sided focalization of a pair of congruences of axes in the direction from the second axes to the first is admitted by those and only those nets for which the sum of the Darboux point invariants is equal to the sum of the tangential invariants:

$$H + K = \overline{H} + \overline{K}. \tag{19}$$

Equalities (15) and (19) will hold simultaneously if

$$H = \overline{H}, \qquad K = \overline{K} \tag{20}$$

or

$$H = \overline{K}, \qquad K = \overline{H}. \tag{21}$$

But equalities (20) characterize nets E, while equalities (21) characterize nets R[5]. We obtain the theorem:

Theorem 2. *For nets E and for nets R, and only for these nets, the pair of congruences of axes is one-sidedly focalizable in the direction from the second axes to the first and forms a T_1 pair in the same direction.*

Remark. Let us note one further characteristic property of nets E, connected with their congruences of axes: in order that a conjugate net be a net E, it is necessary and sufficient that the developable surfaces of the pair of congruences of its axes correspond. This property follows directly from (10), (11), (7), (20).

5. From equations (9) and (13) we conclude that the pair of congruences of axes forms a T_1 pair in the direction from the first axes to the second if

$$\gamma = c\alpha, \qquad \gamma_1 = a\alpha_1, \tag{22}$$

which, by virtue of (7), is equivalent to equalities (21), i.e., this is possible only for nets R.

Since in this case equality (15) holds, the pair of congruences of axes is a T_1 pair also in the opposite direction, i.e., the pair of congruences of axes serves as a pair T. Moreover, it is known [(6)] that if the initial net is a net R, and only in this case, the pair of congruences of axes is two-sidedly focalizable.

Thus, we have the following theorem:

Theorem 3. *The only nets for which the pair of congruences of axes serves as a T_1 pair in the direction from the first axes to the second are nets R. For them the pair of congruences of axes forms a pair T and is two-sidedly focalizable.*

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Note: Figure translations are in progress. See original paper for figures.